Application of superalgebra homology groups to distinguish Engel-like structures

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Abstract

For a long time, tangent bundle of a manifold and the direct sum of multi-vector fields is a prototype of $\mathbb{Z}$-graded superalgebra with the Schouten bracket, and the grading is defined by $a-1$ for a-multivector fields. Recently, we noticed the direct sum of differential forms of a manifold has $\mathbb{Z}$-graded superalgebra structure with a super bracket $[A, B] = (-1)^a d(A \wedge B)$ for a-form A, and the grading of a-form is given by $-a-1$. By Lie derivation, we also see that $\sum_{p=0}^{n} \Lambda^p \mathrm{T}^*(M) \oplus \mathrm{T}(M)$ has $\mathbb{Z}$-graded Lie superalgebra structure, which is a "super" superalgebra of the two superalgebra described above.

For a given $\mathbb{Z}$-graded Lie superalgebra, there is a notion of the weighted (co)chain complex and (co)homology groups. In general, those objects are infinite dimensional and hard to understand the entire properties. If we restrict the manifold above to a finite dimensional Lie group, and multivector fields and differential forms to (left) invariant fields and forms, then (co)chain spaces become finite dimensional. Still studying $\mathbb{Z}$-graded Lie super algebra, which is a generalization of Lie algebra, we encounter complicated manipulation of sign changes depending on each degree. In this note, we introduce our trial of using Maple software to gather possible equations of "Engel like" Lie algebra structures by Jacobi identify, then simply solve them and get six possibilities. Getting weighted chain complex and manipulate its homology groups, we claim that those six possibilities are not isomorphic as Lie algebras in generic.

Keywords

$\mathbb{Z}$-graded Lie super algebra (which is a generalization of Lie algebra), super bracket, the Schouten bracket, chain spaces, the boundary operator, the homology groups, the Betti numbers, and the Euler number.

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1 Introduction

For a 4-dimensional manifold M, if rank 2 distribution D satisfies $D[2] := D + [D,D]$ has rank 3 and $D[3] := D[2] + [D[2], D[2]]$ has rank 4, then the pair (M,D) is called an Engel structure. Here, we apply the above notion for 4-dimensional Lie algebra $\mathfrak{g}$. The pair $(\mathfrak{g}, D)$ is a Engel-type if D is a 2-dimensional subspace of $\mathfrak{g}$, and $\dim D[2] = 3$ where $D[2] := D + [D,D]$, and $\dim D[3] = 4$ where $D[3] := D[2] + [D[2], D[2]]$. From the dimensional restriction, we see a specific basis y_1, y_2 so that $[y_1, y_2] = y_3 \in [D,D]$. We choose y_4 by $[y_1, y_3] = y_4$. Thus,

$$
\begin{array}{llll}
[y_1,y_2] = y_3\,, & [y_2,y_1] = -y_3 & [y_1,y_3] = y_4\,, & [y_3,y_1] = -y_4 \\
[y_1,y_4] = \sum_i C_{1,4,i} y_i\,, & [y_4,y_1] = -\sum_i C_{1,4,i} y_i & [y_2,y_3] = \sum_i C_{2,3,i} y_i\,, & [y_3,y_2] = -\sum_i C_{2,3,i} y_i \\
[y_2,y_4] = \sum_i C_{2,4,i} y_i\,, & [y_4,y_2] = -\sum_i C_{2,4,i} y_i & [y_3,y_4] = \sum_i C_{3,4,i} y_i\,, & [y_4,y_3] = -\sum_i C_{3,4,i} y_i\,.
\end{array}
$$

We let know Maple2021 about the relations above and check Jacobi identities by our maple script `Engel-try-1.mpl`.

```
(* Jun 19, 2022 dim gg = 4; D is a 2dim plane of gg.
D^{2} := D + [D, D]: D^{3} := D^{2} + [ D^{2}, D^{2} ]:
If dim D^{2} = 3 and dim D^{3} = 4, ( gg , D ) is called an Engel structure.
Now want to find concrete Lie algebra having Engel structure and whose Engel's.
*)

C := table():
with(difforms): defform(y=1, C=0):
n := 4: ybase := [seq( y[i],i=1..n) ]: gBase := ybase:
read"Sbt-renew-v1.mpl": # SbtN and those are in BIG.mla
# with(`km/LieAlg/Sbt-renew-v1.mpl`):
kmread("km/LieAlg/Sbt-renew-v1.mpl"):
#For test kmread("X/Y/zz.mpl");

for i to n do SbtT[ y[i], y[i] ] := 0 od:
SbtT[y[1],y[2]] := y[3]: SbtT[y[2],y[1]] := - y[3]:
SbtT[y[1],y[3]] :=  y[4] : SbtT[y[3],y[1]] := - y[4] :

SbtT[y[1],y[4]] := add(C[1,4,j]*y[j] , j=1..n):
SbtT[y[4],y[1]] := -add(C[1,4,j]*y[j] , j=1..n):
myVar := NULL: myVar ,= seq(  C[1,4,j], j=1..n):

for i from 2 to n do for k from i+1 to n do
SbtT[y[i],y[k]] := add( C[i,k,j]*y[j] , j=1..n) :
SbtT[y[k],y[i]] := -add( C[i,k,j]*y[j] , j=1..n) :
myVar ,= seq(  C[i,k,j],  j=1..n): od od:

myJac := proc( A,B,C ) local a,b,c;
SbtN(A, SbtN(B,C)) + SbtN(B, SbtN(C,A)) + SbtN(C, SbtN(A,B)) : end proc:
mySkew  := proc( A,B ) local a,b,c; SbtN(A, B) + SbtN(B, A) : end proc:
(* ukeComm := NULL:
for i to n do for j to n do ukeComm ,= mySkew( y[i], y[j] ) od od: *)

ukeJ := NULL:  for i to n do for j from i+1 to n do for k from j+1 to n do
ukeJ ,= myJac ( y[i], y[j], y[k] ) od od od:
myEqn := NULL:
for i to n do myEqn ,= seq( diff( ukeJ[i], y[j] ), j=1..n ) od:
mySol := solve({myEqn}, {myVar});
```

```
#NG mySolK := map( simplify, mySol ):
mySolK := map( curry( map, simplify ), [ mySol ] ):
mySolTools := SolveTools[PolynomialSystem]({myEqn}, {myVar});
mySolToolsK := map( curry(map, simplify), [mySolTools] ):
for j to nops( mySolK ) do
mySolInd,mySolDep := selectremove( i-> lhs(i)= rhs(i), mySolK[j] ):
ourSolInd[j] := map( lhs, mySolInd ):
ourSolDep[j] := mySolDep  :
mySolToolInd, mySolToolDep := selectremove( i-> lhs(i)= rhs(i), mySolToolsK[j] ):
ourSolToolInd[j] := map( lhs, mySolToolInd ):
ourSolToolDep[j] := mySolToolDep:
od:
(* UKE := table(sparse=NULL): SbtTT := table():
for i to nops([mySol]) do print(i,"", mySol[i]):
for j to n do for k from j+1 to n do
    UKE[i] ,= SbtTT [y[j],y[k]] = simplify( SbtT[y[j],y[k]], mySol[i] ) od od:
print(UKE[i]): od: *)
# assign( UKE[1] ): BktT[ y[i], y[j] ]:
# save mySol, SbtT, UKE, "Engel-try-1-out2.txt":
# save ourSolToolInd, ourSolToolDep, ourSolInd, ourSolDep, SbtT,
"Engel-try-1-out4.txt":
(* Important caution. ourSol and ourSotTool are not exactly equal. 1 to 4 are
equal but
5 and 6 are swapped. So, we do use one way,  do not mixed using.  Jul 14, 2022 *)
(* i := 1: while( X[i] <> [] ) do some job : i := i+1 od:  *)
```

We show how Maple works well.

```
> restart:read"Engel-try-1.mpl":
> (*Jacobi id *): ukeJ;
```

$$
\begin{aligned}
& C_{2,3,2}y_3 + C_{2,3,3}y_4 + C_{2,3,4}(C_{1,4,1}y_1 + C_{1,4,2}y_2 + C_{1,4,3}y_3 + C_{1,4,4}y_4) - C_{2,4,1}y_1 \\
& \quad - C_{2,4,2}y_2 - C_{2,4,3}y_3 - C_{2,4,4}y_4, C_{2,4,2}y_3 + C_{2,4,3}y_4 + C_{2,4,4}(C_{1,4,1}y_1 + C_{1,4,2}y_2 \\
& \quad + C_{1,4,3}y_3 + C_{1,4,4}y_4) + C_{1,4,1}y_3 - C_{1,4,3}(C_{2,3,1}y_1 + C_{2,3,2}y_2 + C_{2,3,3}y_3 \\
& \quad + C_{2,3,4}y_4) - C_{1,4,4}(C_{2,4,1}y_1 + C_{2,4,2}y_2 + C_{2,4,3}y_3 + C_{2,4,4}y_4) - C_{3,4,1}y_1 \\
& \quad - C_{3,4,2}y_2 - C_{3,4,3}y_3 - C_{3,4,4}y_4, C_{3,4,2}y_3 + C_{3,4,3}y_4 + C_{3,4,4}(C_{1,4,1}y_1 + C_{1,4,2}y_2 \\
& \quad + C_{1,4,3}y_3 + C_{1,4,4}y_4) + C_{1,4,1}y_4 - C_{1,4,2}(-C_{2,3,1}y_1 - C_{2,3,2}y_2 - C_{2,3,3}y_3 \\
& \quad - C_{2,3,4}y_4) - C_{1,4,4}(C_{3,4,1}y_1 + C_{3,4,2}y_2 + C_{3,4,3}y_3 + C_{3,4,4}y_4), -C_{3,4,1}y_3 \\
& \quad + C_{3,4,3}(C_{2,3,1}y_1 + C_{2,3,2}y_2 + C_{2,3,3}y_3 + C_{2,3,4}y_4) + C_{3,4,4}(C_{2,4,1}y_1 + C_{2,4,2}y_2 \\
& \quad + C_{2,4,3}y_3 + C_{2,4,4}y_4) + C_{2,4,1}y_4 - C_{2,4,2}(-C_{2,3,1}y_1 - C_{2,3,2}y_2 - C_{2,3,3}y_3 \\
& \quad - C_{2,3,4}y_4) - C_{2,4,4}(C_{3,4,1}y_1 + C_{3,4,2}y_2 + C_{3,4,3}y_3 + C_{3,4,4}y_4) + C_{2,3,1}(\\
& \quad -C_{1,4,1}y_1 - C_{1,4,2}y_2 - C_{1,4,3}y_3 - C_{1,4,4}y_4) + C_{2,3,2}(-C_{2,4,1}y_1 - C_{2,4,2}y_2 \\
& \quad - C_{2,4,3}y_3 - C_{2,4,4}y_4) + C_{2,3,3}(-C_{3,4,1}y_1 - C_{3,4,2}y_2 - C_{3,4,3}y_3 - C_{3,4,4}y_4)
\end{aligned} \tag{1}
$$

```
> restart:read"Engel-try-1.mpl":
> (*Jacobi id *): ukeJ:
> (* eqns from Jacobi id *): myEqn;
```

$$C_{2,3,4}C_{1,4,1}-C_{2,4,1}, C_{2,3,4}C_{1,4,2}-C_{2,4,2}, C_{2,3,4}C_{1,4,3}+C_{2,3,2}-C_{2,4,3}, C_{2,3,4}C_{1,4,4}+C_{2,3,3}-C_{2,4,4}, C_{2,4,4}C_{1,4,1}-C_{1,4,3}C_{2,3,1}-C_{1,4,4}C_{2,4,1}-C_{3,4,1}, C_{2,4,4}C_{1,4,2}-C_{1,4,3}C_{2,3,2}-C_{1,4,4}C_{2,4,2}-C_{3,4,2}, -C_{1,4,3}C_{2,3,3}+C_{2,4,4}C_{1,4,3}-C_{1,4,4}C_{2,4,3}+C_{1,4,1}+C_{2,4,2}-C_{3,4,3}, -C_{2,3,4}C_{1,4,3}+C_{2,4,3}-C_{3,4,4}, C_{3,4,4}C_{1,4,1}+C_{1,4,2}C_{2,3,1}-C_{1,4,4}C_{3,4,1}, C_{1,4,2}C_{2,3,2}+C_{3,4,4}C_{1,4,2}-C_{1,4,4}C_{3,4,2}, C_{1,4,2}C_{2,3,3}+C_{3,4,4}C_{1,4,3}-C_{1,4,4}C_{3,4,3}+C_{3,4,2}, C_{2,3,4}C_{1,4,2}+C_{1,4,1}+C_{3,4,3}, -C_{2,3,1}C_{1,4,1}+C_{2,4,2}C_{2,3,1}+C_{3,4,3}C_{2,3,1}-C_{2,3,2}C_{2,4,1}-C_{2,3,3}C_{3,4,1}+C_{3,4,4}C_{2,4,1}-C_{2,4,4}C_{3,4,1}, -C_{1,4,2}C_{2,3,1}+C_{3,4,3}C_{2,3,2}-C_{2,3,3}C_{3,4,2}+C_{3,4,4}C_{2,4,2}-C_{2,4,4}C_{3,4,2}, -C_{1,4,3}C_{2,3,1}-C_{2,3,2}C_{2,4,3}+C_{2,4,2}C_{2,3,3}+C_{3,4,4}C_{2,4,3}-C_{2,4,4}C_{3,4,3}-C_{3,4,1}, -C_{2,3,1}C_{1,4,4}-C_{2,3,2}C_{2,4,4}-C_{2,3,3}C_{3,4,4}+C_{2,4,2}C_{2,3,4}+C_{3,4,3}C_{2,3,4}+C_{2,4,1} \tag{1}$$

```
> restart:read"Engel-try-1.mpl":
> (* solve( {myEqn}, {myVar}) *): print( ourSolInd):
```

$$\mathrm{table}\left(\left[1=\{C_{1,4,3}, C_{1,4,4}, C_{2,3,4}, C_{2,4,4}\}, 2=\{C_{1,4,3}, C_{1,4,4}, C_{2,3,4}, C_{2,4,4}\}, 3=\{C_{1,4,2}, C_{1,4,3}, C_{1,4,4}, C_{2,4,4}\}, 4=\{C_{2,3,1}, C_{2,3,4}, C_{2,4,4}\}, 5=\{C_{1,4,2}, C_{1,4,3}, C_{2,3,4}\}, 6=\{C_{1,4,3}, C_{2,3,1}, C_{2,3,4}, C_{3,4,4}\}\right]\right) \tag{1}$$

```
> restart:read"Engel-try-1.mpl":
> print(ourSolDep[1]):
```

$$\{C_{1,4,1}=0, C_{1,4,2}=0, C_{2,3,1}=0, C_{2,3,2}=0, C_{2,3,3}=-C_{2,3,4}C_{1,4,4}+C_{2,4,4}, C_{2,4,1}=0, C_{2,4,2}=0, C_{2,4,3}=C_{2,3,4}C_{1,4,3}, C_{3,4,1}=0, C_{3,4,2}=0, C_{3,4,3}=0, C_{3,4,4}=0\} \tag{1}$$

```
> restart:read"Engel-try-1.mpl":
> for i from 3 to nops( [ indices(ourSolDep) ]) do print(
  ourSolDep[i]) od:
```

$$\left\{C_{1,4,1}=-\frac{C_{1,4,2}C_{2,4,4}}{C_{1,4,4}}, C_{2,3,1}=0, C_{2,3,2}=0, C_{2,3,3}=0, C_{2,3,4}=\frac{C_{2,4,4}}{C_{1,4,4}}, C_{2,4,1}=-\frac{C_{1,4,2}C_{2,4,4}^2}{C_{1,4,4}^2}, C_{2,4,2}=\frac{C_{1,4,2}C_{2,4,4}}{C_{1,4,4}}, C_{2,4,3}=\frac{C_{1,4,3}C_{2,4,4}}{C_{1,4,4}}, C_{3,4,1}=0, C_{3,4,2}=0, C_{3,4,3}=0, C_{3,4,4}=0\right\}$$

$$\{C_{1,4,1}=0, C_{1,4,2}=0, C_{1,4,3}=0, C_{1,4,4}=0, C_{2,3,2}=0, C_{2,3,3}=C_{2,4,4}, C_{2,4,1}=0, C_{2,4,2}=0, C_{2,4,3}=0, C_{3,4,1}=0, C_{3,4,2}=0, C_{3,4,3}=0, C_{3,4,4}=0\}$$

$$\{C_{1,4,1}=-C_{2,3,4}C_{1,4,2}, C_{1,4,4}=0, C_{2,3,1}=0, C_{2,3,2}=0, C_{2,3,3}=0, C_{2,4,1}=-C_{1,4,2}C_{2,3,4}^2, C_{2,4,2}=C_{2,3,4}C_{1,4,2}, C_{2,4,3}=C_{2,3,4}C_{1,4,3}, C_{2,4,4}=0, C_{3,4,1}=0, C_{3,4,2}=0, C_{3,4,3}=0, C_{3,4,4}=0\}$$

$$\{C_{1,4,1}=0, C_{1,4,2}=0, C_{1,4,4}=0, C_{2,3,2}=C_{3,4,4}, C_{2,3,3}=0, C_{2,4,1}=0, C_{2,4,2}=0, C_{2,4,3}=C_{2,3,4}C_{1,4,3}+C_{3,4,4}, C_{2,4,4}=0, C_{3,4,1}=-C_{1,4,3}C_{2,3,1}, C_{3,4,2}=-C_{3,4,4}C_{1,4,3}, C_{3,4,3}=0\} \tag{1}$$

```
> restart:read"Engel-try-1.mpl":
> pprint(convert(ourSolDep[2], list)):
```

$$C_{1,4,1} = -\frac{\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - 2\,C_{2,4,4}\right)}{8}$$

$$C_{1,4,2} = -\frac{1}{8}\,C_{1,4,4}^3 - \frac{1}{2}\,C_{1,4,3}\,C_{1,4,4}$$

$$C_{2,3,1} = -\frac{\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)\left(C_{2,3,4}\,C_{1,4,4} - 2\,C_{2,4,4}\right)}{2\,C_{1,4,4}^2}$$

$$C_{2,3,2} = -\frac{\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)}{2\,C_{1,4,4}}$$

$$C_{2,3,3} = -C_{2,3,4}\,C_{1,4,4} + C_{2,4,4}$$

$$C_{2,4,1} = -\frac{C_{2,3,4}\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - 2\,C_{2,4,4}\right)}{8}$$

$$C_{2,4,2} = -\frac{C_{1,4,4}\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)C_{2,3,4}}{8}$$

$$C_{2,4,3} = \frac{-C_{1,4,4}^3\,C_{2,3,4} - 2\,C_{1,4,3}\,C_{1,4,4}\,C_{2,3,4} + C_{2,4,4}\,C_{1,4,4}^2 + 4\,C_{2,4,4}\,C_{1,4,3}}{2\,C_{1,4,4}}$$

$$C_{3,4,1} = \frac{2\left(\frac{C_{1,4,4}^2}{4} + C_{1,4,3}\right)^2\left(C_{2,3,4}\,C_{1,4,4} - 2\,C_{2,4,4}\right)\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)}{C_{1,4,4}^2}$$

$$C_{3,4,2} = \frac{2\left(\frac{C_{1,4,4}^2}{4} + C_{1,4,3}\right)^2\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)}{C_{1,4,4}}$$

$$C_{3,4,3} = \frac{\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)}{4}$$

$$C_{3,4,4} = -\frac{\left(C_{1,4,4}^2 + 4\,C_{1,4,3}\right)\left(C_{2,3,4}\,C_{1,4,4} - C_{2,4,4}\right)}{2\,C_{1,4,4}} \qquad \textbf{(1)}$$

As you expect, in the original `Engel-try-1.mpl` if we declare `save` at the almost end of the file, we get an output file `Engel-try-1-out4.txt`. Preparing the general form of Lie bracket candidate of Engel-type, we gather Jacobi conditions `myEqn`. Solving them, we have 6 cases, and denote them as `ourSolInd[i]` and `ourSolDep[i]` for $i = 1, \ldots, 6$. As you see, the independent variable $C_{1,4,4}$ of the second and third cases is assumed to be non-zero. `ourSolDep[2]` looks rather complicated comparing the others. Our main purpose of this note is to distinguish those 6 cases are Lie algebra isomorphic or not by watching their homology groups in superalgebra sense. Our strategy is simple, that is, if two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic as Lie algebras, their superalgebras are isomorphic, and so weighted homology groups coincide for each primary weight and the superalgebra type. Contrary, if the corresponding homology groups are not equal for some same type of superalgebras and for some primary weight, then the original Lie algebras are not isomorphic.

Weight 0, Common Headers $\begin{bmatrix} \text{m} & 0 & 1 & 2 & 3 & 4 \\ \text{SpaD} & 1 & 4 & 6 & 4 & 1 \end{bmatrix}$	Weight 1, Common Headers $\begin{bmatrix} \text{m} & 1 & 2 & 3 & 4 & 5 \\ \text{SpaD} & 6 & 24 & 36 & 24 & 6 \end{bmatrix}$	Weight 0, Common Headers $\begin{bmatrix} \text{m} & 1 & 2 & 3 & 4 & 5 & 6 \\ \text{SpaD} & 4 & 37 & 108 & 142 & 88 & 21 \end{bmatrix}$
Weight 0 of Engel ourType 1 $\begin{bmatrix} \text{KerD} & 1 & 4 & 4 & 1 & 0 \\ \text{Bett} & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$	Weight 1 of Engel ourType 1 $\begin{bmatrix} \text{KerD} & 6 & 19 & 19 & 6 & 0 \\ \text{Bett} & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 1 $\begin{bmatrix} \text{KerD} & 4 & 33 & 76 & 68 & 21 & 0 \\ \text{Bett} & 0 & 1 & 2 & 1 & 0 & 0 \end{bmatrix}$
Weight 0 of Engel ourType 2 $\begin{bmatrix} \text{KerD} & 1 & 4 & 3 & 1 & 0 \\ \text{Bett} & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$	Weight 1 of Engel ourType 2 $\begin{bmatrix} \text{KerD} & 6 & 18 & 18 & 6 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 2 $\begin{bmatrix} \text{KerD} & 4 & 33 & 75 & 67 & 21 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Weight 0 of Engel ourType 3 $\begin{bmatrix} \text{KerD} & 1 & 4 & 3 & 1 & 0 \\ \text{Bett} & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$	Weight 1 of Engel ourType 3 $\begin{bmatrix} \text{KerD} & 6 & 18 & 18 & 6 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 3 $\begin{bmatrix} \text{KerD} & 4 & 33 & 75 & 67 & 21 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Weight 0 of Engel ourType 4 $\begin{bmatrix} \text{KerD} & 1 & 4 & 3 & 1 & 0 \\ \text{Bett} & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$	Weight 1 of Engel ourType 4 $\begin{bmatrix} \text{KerD} & 6 & 18 & 18 & 6 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 4 $\begin{bmatrix} \text{KerD} & 4 & 33 & 75 & 67 & 21 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
Weight 0 of Engel ourType 5 $\begin{bmatrix} \text{KerD} & 1 & 4 & 3 & 1 & 1 \\ \text{Bett} & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$	Weight 1 of Engel ourType 5 $\begin{bmatrix} \text{KerD} & 6 & 18 & 18 & 6 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 5 $\begin{bmatrix} \text{KerD} & 4 & 33 & 77 & 67 & 23 & 1 \\ \text{Bett} & 0 & 2 & 2 & 2 & 3 & 1 \end{bmatrix}$
Weight 0 of Engel ourType 6 $\begin{bmatrix} \text{KerD} & 1 & 4 & 3 & 1 & 1 \\ \text{Bett} & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$	Weight 1 of Engel ourType 6 $\begin{bmatrix} \text{KerD} & 6 & 18 & 18 & 6 & 0 \\ \text{Bett} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Weight 2 of Engel ourType 6 $\begin{bmatrix} \text{KerD} & 4 & 33 & 77 & 67 & 21 & 2 \\ \text{Bett} & 0 & 2 & 2 & 0 & 2 & 2 \end{bmatrix}$

Table 1: by Engel-mySols-v2.mpl

2 Application of super homology of tangent type $\overset{4}{\underset{\ell=1}{\oplus}} \Lambda^{\ell}\mathfrak{g}$

We have six types after solving the Jacobi identity conditions. In the type 2 and 3, some coefficients are fractions of denominator $C_{1,4,4}$, this means it is non-zero. Looking at those six types, the type 2 seems to be more complicated. We have a super homology group theory for superalgebra, especially derived from Lie algebras which come from Engel like structure as we got above. We ask Maple to manipulate superalgebra homology groups with the primary weight 0, 1, or 2. In the following, we show the outputs by Maple for the six types in "generic". In the weight 0 case, the super homology group is just the Lie algebra homology group and it is natural to treat 0-th chain space $\mathfrak{g}^0 = \mathbb{R}$, and modify the output by Maple and we conclude the Euler number is 0 for the weight 0 case, too.
Then we have the proposition below.

Proposition 2.1 By the three weighted generic results about Betti numbers for each type `Type(i)` where $i = 1 \ldots 6$, those six types are divided into 4 classes Type(1), {Type(2), Type(3), Type(4)}, Type(4) and Type(6), so that each two Lie algebras are not isomorphic if they belong different classes.

Remark 2.1 We expect to find more refinement if we try the third or more weight cases. The word "generic" in the theorem above means we just follow Maple formal manipulations. When we want the kernel dimension, we have two ways: one is solve the linear equations, and the number of free parameters means the kernel dimension. The other is to fix the Groebner basis whose number shows the rank. For instance, assume we have a linear system $\{ax + by = 0\}$ for x, y are independent variables and a, b are general constants. Maple `solve`-command shows $x = -\frac{by}{a}, y = y$ and Maple `Groebner:-Basis({a x + b y}, tdeg( x,y ))` shows $[ax + by]$ formally. We say those answers are "generic". If $a \neq 0$ or $b \neq 0$, then the result is included in the generic solution. But, if

$a=b=0$, then the kernel space is 2-dimensional and the Groebner basis is 0-dimensional. We may be careful Maple does not care about this case.

3 Application of super homology of cotangent type $\overset{4}{\underset{\ell=0}{\oplus}} \Lambda^{\ell}\mathfrak{g}^{*}$

We have superalgebras on the exterior algebra of differential forms of manifold . We restrict our attention to the invariant forms of Lie groups, and have the super homology groups. We refer to [8] about precise definitions, but in this note, the grading for forms is sharp, namely, the grade of p-form is $-(p+1)$. On the contrary, the grade of p-multi vector field is $p-1$. Given a (primary) weight wt, if wt is non-negative, then the chain space is empty. Otherwise we have chain space bounded to $|wt|$, which is 1-dimensional, consists of $\underbrace{1 \triangle \cdots \triangle 1}_{|wt|}$, which is a kernel element.

m	1	2	3	4	5
SpaceDim	1	28	12	4	1
KerDim	1				1
Betti					1

We concentrate for the weight -5 here. We follow the definition of m -th chain space. It consists of $\mathfrak{h}_{-1}^{a_1} \wedge \mathfrak{h}_{-2}^{a_2} \wedge \mathfrak{h}_{-3}^{a_3} \wedge \mathfrak{h}_{-4}^{a_4} \wedge \mathfrak{h}_{-5}^{a_5}$ where $a_1+\cdots+a_5=m$ and $(-1)a_1+\cdots+(-5)a_5=-5$. Directly, we see that $m<=5$ and a_i comes from the Young diagrams of area 5 and length m. Thus the common table becomes as seen on the right end. When $m=5$, the chain space is $\mathfrak{h}_{-1}^{5}$ and 1-dimensional. When $m=4$, the chain space is $\mathfrak{h}_{-1}^{3} \triangle \mathfrak{h}_{-2}$ and 4-dimensional. When $m=3$, the chain space is $(\mathfrak{h}_{-1} \triangle \mathfrak{h}_{-2}^{2}) \oplus (\mathfrak{h}_{-1}^{2} \triangle \mathfrak{h}_{-3})$ and 12-dimensional. When $m=2$, the chain space is $(\mathfrak{h}_{-2} \triangle \mathfrak{h}_{-3}) \oplus (\mathfrak{h}_{-1} \triangle \mathfrak{h}_{-4})$ and 28-dimensional. When $m=1$, the chain space is $\mathfrak{h}_{-5} = \&\wedge(z_1,z_2,z_3,z_4)$ and 1-dimensional. By manipulating weight (-5) super homology groups below, we classify 6 types into 3 as Type1, Type2 – Type4, and Type5 – Type 6, which is not new in computation of the previous section.

```
> restart:read"DualAlg-Engel-v44.mpl":
> print("Weight",wt,"Common Headers."): (SubMat[0,wt]);
```

"Weight", 5, "Common Headers."

$$\begin{bmatrix} \text{"m"} & 1 & 2 & 3 & 4 & 5 \\ \text{"SpaD"} & 1 & 28 & 12 & 4 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 1: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 1

["dz[1]=", 0]

["dz[2]=", 0]

$$\left[\text{"dz[3]="}, -z_1 \,\&\hat{}\, z_2 - C_{1,4,3} z_1 \,\&\hat{}\, z_4 + \left(C_{1,4,4} C_{2,3,4} - C_{2,4,4}\right) z_2 \,\&\hat{}\, z_3 - C_{2,3,4} C_{1,4,3} z_2 \,\&\hat{}\, z_4\right]$$

$$\left[\text{"dz[4]="}, -z_1 \,\&\hat{}\, z_3 - C_{1,4,4} z_1 \,\&\hat{}\, z_4 - C_{2,3,4} z_2 \,\&\hat{}\, z_3 - C_{2,4,4} z_2 \,\&\hat{}\, z_4\right]$$

$$\begin{bmatrix} \text{"KerD"} & 1 & 27 & 4 & 2 & 1 \\ \text{"Bett"} & 0 & 19 & 2 & 2 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 2: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 2

"Output of d(z[j]) are too long, so skip them."

$$\begin{bmatrix} \text{"KerD"} & 1 & 27 & 3 & 1 & 1 \\ \text{"Bett"} & 0 & 18 & 0 & 1 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 3: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 3

$$\left[\text{"dz[1]=",}\ \frac{C_{1,4,2}\,C_{2,4,4}\,z_1 \,\&\hat{}\, z_4}{C_{1,4,4}} + \frac{C_{1,4,2}\,C_{2,4,4}^2\,z_2 \,\&\hat{}\, z_4}{C_{1,4,4}^2}\right]$$

$$\left[\text{"dz[2]=",}\ -C_{1,4,2}\,z_1 \,\&\hat{}\, z_4 - \frac{C_{1,4,2}\,C_{2,4,4}\,z_2 \,\&\hat{}\, z_4}{C_{1,4,4}}\right]$$

$$\left[\text{"dz[3]=",}\ -z_1 \,\&\hat{}\, z_2 - C_{1,4,3}\,z_1 \,\&\hat{}\, z_4 - \frac{z_2 \,\&\hat{}\, z_4\,C_{1,4,3}\,C_{2,4,4}}{C_{1,4,4}}\right]$$

$$\left[\text{"dz[4]=",}\ -z_1 \,\&\hat{}\, z_3 - C_{1,4,4}\,z_1 \,\&\hat{}\, z_4 - \frac{C_{2,4,4}\,z_2 \,\&\hat{}\, z_3}{C_{1,4,4}} - C_{2,4,4}\,z_2 \,\&\hat{}\, z_4\right]$$

$$\begin{bmatrix} \text{"KerD"} & 1 & 27 & 3 & 1 & 1 \\ \text{"Bett"} & 0 & 18 & 0 & 1 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 4: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 4

$$\left[\text{"dz[1]=",}\ -C_{2,3,1}\,z_2 \,\&\hat{}\, z_3\right]$$

$$\left[\text{"dz[2]=",}\ 0\right]$$

$$\left[\text{"dz[3]=",}\ -z_1 \,\&\hat{}\, z_2 - C_{2,4,4}\,z_2 \,\&\hat{}\, z_3\right]$$

$$\left[\text{"dz[4]=",}\ -z_1 \,\&\hat{}\, z_3 - C_{2,3,4}\,z_2 \,\&\hat{}\, z_3 - C_{2,4,4}\,z_2 \,\&\hat{}\, z_4\right]$$

$$\begin{bmatrix} \text{"KerD"} & 1 & 27 & 3 & 1 & 1 \\ \text{"Bett"} & 0 & 18 & 0 & 1 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 5: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 5

$$\left[\text{"dz[1]=",}\ C_{2,3,4}\,C_{1,4,2}\,z_1 \,\&\hat{}\, z_4 + C_{1,4,2}\,C_{2,3,4}^2\,z_2 \,\&\hat{}\, z_4\right]$$

$$\left[\text{"dz[2]=",}\ -C_{1,4,2}\,z_1 \,\&\hat{}\, z_4 - C_{2,3,4}\,C_{1,4,2}\,z_2 \,\&\hat{}\, z_4\right]$$

$$\left[\text{"dz[3]=",}\ -z_1 \,\&\hat{}\, z_2 - C_{1,4,3}\,z_1 \,\&\hat{}\, z_4 - C_{2,3,4}\,C_{1,4,3}\,z_2 \,\&\hat{}\, z_4\right]$$

$$\left[\text{"dz[4]=",}\ -z_1 \,\&\hat{}\, z_3 - C_{2,3,4}\,z_2 \,\&\hat{}\, z_3\right]$$

$$\begin{bmatrix} \text{"KerD"} & 1 & 28 & 3 & 1 & 1 \\ \text{"Bett"} & 1 & 19 & 0 & 1 & 1 \end{bmatrix} \quad \textbf{(1)}$$

```
> restart:read"DualAlg-Engel-v44.mpl":
> j := 6: Rinji(j,wt): SubMat[j,wt];
```

"Weight", 5, "of Engel Type", 6

$$["dz[1]=", -C_{2,3,1}\,z_2 \,\&\hat{}\, z_3 + C_{1,4,3}\,C_{2,3,1}\,z_3 \,\&\hat{}\, z_4]$$

$$["dz[2]=", -C_{3,4,4}\,z_2 \,\&\hat{}\, z_3 + C_{3,4,4}\,C_{1,4,3}\,z_3 \,\&\hat{}\, z_4]$$

$$["dz[3]=", -z_1 \,\&\hat{}\, z_2 - C_{1,4,3}\,z_1 \,\&\hat{}\, z_4 + \left(-C_{2,3,4}\,C_{1,4,3} - C_{3,4,4}\right) z_2 \,\&\hat{}\, z_4]$$

$$["dz[4]=", -z_1 \,\&\hat{}\, z_3 - C_{2,3,4}\,z_2 \,\&\hat{}\, z_3 - C_{3,4,4}\,z_3 \,\&\hat{}\, z_4]$$

$$\begin{bmatrix} \text{"KerD"} & 1 & 28 & 3 & 1 & 1 \\ \text{"Bett"} & 1 & 19 & 0 & 1 & 1 \end{bmatrix} \qquad \textbf{(1)}$$

Remark 3.1 Here, we explain about "generic" which was stated in the theorem 2.1 before, by outputs in this section more precisely. In general, we follow both simple `solve` and `Groebner:-Basis`. An easy checkpoint is `denominator` or possibility of enumeration of elements of `Groebner:-Basis`.

`Type(1):  The case of` $m=2$:

$$SOL = q_1 = q_1, \ldots, q_{11} = q_{11}, q_{13} = q_{13}, \ldots, q_{28} = q_{28};$$
$$q_{12} = -(C_{2,3,4}q_6 - C_{2,3,4}q_{15} + C_{2,3,4}q_{20} - C_{2,3,4}q_{27} - q_{17} + q_{22} - q_{28} - \frac{2}{C_{1,4,4}}(-C_{2,4,4}q_6 + C_{2,4,4}q_{15} - C_{2,4,4}q_{20} + C_{2,4,4}q_{27}))$$

$$EGB = [q_6(C_{1,4,4}C_{2,3,4} - 2C_{2,4,4}) + q_{12}C_{1,4,4} + (-C_{1,4,4}C_{2,3,4} + 2C_{2,4,4})q_{15} - q_{17}C_{1,4,4} + q_{20}(C_{1,4,4}C_{2,3,4} - 2C_{2,4}$$
$$+ q_{22}C_{1,4,4} + (-C_{1,4,4}C_{2,3,4} + 2C_{2,4,4})q_{27} - q_{28}C_{1,4,4}]:$$

$C_{1,4,4} \neq 0$ is in generic. If $C_{1,4,4} = 0$ then ORG = EGB = $[-2C_{2,4,4}(q_6 - q_{15} + q_{20} - q_{27})\ ]$, and so if $C_{2,4,4} \neq 0$ then the case is in generic. If $C_{1,4,4} = 0$ and $C_{2,4,4} = 0$, then rank is 0 and the kernel dim is 28.

`Type(1):  The case of` $m=3$:

$$SOL = \{q_1 = q_1, q_2 = 0, q_3 = 0, q_4 = 0, q_5 = 0, q_6 = 0, q_7 = q_7, q_8 = q_8, q_9 = q_9,$$
$$q_{10} = (C_{1,4,4}^2C_{2,3,4}q_8 + C_{1,4,3}C_{2,3,4}q_8 - C_{1,4,4}C_{2,3,4}q_9 - C_{1,4,4}C_{2,4,4}q_8 + C_{2,4,4}q_9)/C_{1,4,3},$$
$$q_{11} = -C_{1,4,4}C_{2,3,4}q_8 + C_{2,3,4}q_9 + C_{2,4,4}q_8, q_{12} = 0\};$$

$$EGB = [q_{12}, q_9(C_{1,4,3}C_{2,3,4}^2 + C_{1,4,4}C_{2,3,4}C_{2,4,4} - C_{2,4,4}^2) + q_{10}(-C_{1,4,3}C_{1,4,4}C_{2,3,4} + C_{1,4,3}C_{2,4,4})$$
$$+ q_{11}(-C_{1,4,4}^2C_{2,3,4} - C_{1,4,3}C_{2,3,4} + C_{1,4,4}C_{2,4,4}),$$
$$(C_{1,4,3}C_{2,3,4}^2 + C_{1,4,4}C_{2,3,4}C_{2,4,4} - C_{2,4,4}^2)q_8 - C_{1,4,3}C_{2,3,4}q_{10} + (-C_{1,4,4}C_{2,3,4} + C_{2,4,4})q_{11}, q_6, q_5, q_4, q_3, q_2];$$

Thus, if $C_{1,4,3} \neq 0$ then it is generic and rank is 8. If $C_{1,4,3} = 0$, then $EGB = [q_{12}, A, B, q_6, q_5, q_4, q_3, q_2]$ where

$$A = (C_{1,4,4}C_{2,3,4} - C_{2,4,4})(C_{2,4,4}q_9 - C_{1,4,4}q_{11}), \quad \text{and} \quad B = (C_{1,4,4}C_{2,3,4} - C_{2,4,4})(C_{2,4,4}q_8 - q_{11}).$$

If $C_{1,4,4}C_{2,3,4} - C_{2,4,4} = 0$, $A = B = 0$ and the rank is 6. If $C_{1,4,4}C_{2,3,4} - C_{2,4,4} \neq 0$, then $B \neq 0$ and $A' = C_{2,4,4}r_9 - C_{1,4,4}r_{11}$ $B' = C_{2,4,4}r_8 - r_{11}$. The wedge product $\&\hat{}(A', B') = -C_{2,4,4}(C_{2,4,4}\&\hat{}(r_8, r_9) - C_{1,4,4}\&\hat{}(r_8, r_{10}) + \&\hat{}(r_9, r_{10})$. Thus, if $C_{2,4,4} = 0$ then $A = C_{1,4,4}B$ and the rank is 7. If $C_{2,4,4} \neq 0$ then A and B are linearly independent and the rank is 8 (in generic).

Weight 6, Common Headers

$$\begin{bmatrix} \text{m} & 2 & 3 & 4 & 5 & 6 \\ \text{SpaD} & 38 & 32 & 12 & 4 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 1

$$\begin{bmatrix} \text{KerD} & 38 & 12 & 4 & 2 & 1 \\ \text{Bett} & 18 & 4 & 2 & 2 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 2

$$\begin{bmatrix} \text{KerD} & 38 & 10 & 3 & 1 & 1 \\ \text{Bett} & 16 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 3

$$\begin{bmatrix} \text{KerD} & 38 & 12 & 3 & 1 & 1 \\ \text{Bett} & 18 & 3 & 0 & 1 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 4

$$\begin{bmatrix} \text{KerD} & 38 & 10 & 3 & 1 & 1 \\ \text{Bett} & 16 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 5

$$\begin{bmatrix} \text{KerD} & 6 & 13 & 3 & 1 & 1 \\ \text{Bett} & 19 & 4 & 0 & 1 & 1 \end{bmatrix}$$

Weight 6 of Engel ourType 6

$$\begin{bmatrix} \text{KerD} & 38 & 11 & 3 & 1 & 1 \\ \text{Bett} & 17 & 2 & 0 & 1 & 1 \end{bmatrix}$$

Weight 7, Common Headers

$$\begin{bmatrix} \text{m} & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{SpaD} & 28 & 76 & 32 & 12 & 4 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 1

$$\begin{bmatrix} \text{KerD} & 28 & 50 & 10 & 4 & 2 & 1 \\ \text{Bett} & 4 & 28 & 2 & 2 & 2 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 2

$$\begin{bmatrix} \text{KerD} & 28 & 50 & 9 & 3 & 1 & 1 \\ \text{Bett} & 4 & 27 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 3

$$\begin{bmatrix} \text{KerD} & 28 & 52 & 9 & 3 & 1 & 1 \\ \text{Bett} & 6 & 29 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 4

$$\begin{bmatrix} \text{KerD} & 28 & 50 & 9 & 3 & 1 & 1 \\ \text{Bett} & 4 & 27 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 5

$$\begin{bmatrix} \text{KerD} & 28 & 53 & 10 & 3 & 1 & 1 \\ \text{Bett} & 7 & 31 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Weight 7 of Engel ourType 6

$$\begin{bmatrix} \text{KerD} & 28 & 53 & 10 & 3 & 1 & 1 \\ \text{Bett} & 7 & 31 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Table 2: by DualAlg-Engel-v44.mpl

`Type(1): Other cases are in generic.`

By the same discussion, we have checked the types 2, 3, 5 and 6 are only generic. For the types 2 and 3, $C_{1,4,4} \neq 0$ is assumed.

`Type(4): The case of` $m = 2$: The kernel condition is $[-2C_{2,4,4}(q_6 - q_{15} + q_{20} - q_{27})]$. So if $C_{2,4,4} \neq 0$ then the kernel dimension is 27 and the rank is 1 (in generic). If $C_{2,4,4} = 0$ then the kernel dimension is 28 and the rank is 0.

`Type(4): Other cases are in generic.`

4 Cases for the extended superalgebra $\overset{4}{\underset{\ell=0}{\oplus}} \Lambda^{\ell}\mathfrak{g}^* \oplus \mathfrak{g}$

In [8], we know that $\mathfrak{g} \oplus \overset{4}{\underset{\ell=0}{\oplus}} \Lambda^{\ell}\mathfrak{g}^*$ is a super superalgebra of $\overset{4}{\underset{\ell=0}{\oplus}} \Lambda^{\ell}\mathfrak{g}^*$ by using Lie derivation, in more general context. In this section, we try and see the homology groups of those extended superalgebras for each Engel-like Lie algebra $\mathfrak{g}$. The following output by Maple2021, which are drove by ZeroPlus-Engel-v3.mpl and are shown below, implies that six types are divided into 5 classes. Right now, type 2 and type 4 have the same table and are not distinguished on this job. However, using other enhanced superalgebras with the weight -3 in this section, we conclude those six types are not isomorphic as Lie algebras. Thus, we claim the next theorem.

Theorem 4.1 The six types, which come from Lie algebras axioms, are mutually not isomorphic in generic.

Remark 4.1 When the weight is -2, the output on the left hand side below says nothing about Type 2 and Type 4. The weight -3 output on the right hand side claim that six types are mutually not isomorphic as Lie algebras.

Weight 2, Common Headers

$$\begin{bmatrix} \text{m} & 1 & 2 & 3 & 4 & 5 & 6 \\ \text{SpaD} & 4 & 17 & 28 & 22 & 8 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 1

$$\begin{bmatrix} \text{KerD} & 4 & 13 & 16 & 10 & 4 & 1 \\ \text{Bett} & 0 & 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 2

$$\begin{bmatrix} \text{KerD} & 4 & 13 & 17 & 10 & 4 & 1 \\ \text{Bett} & 0 & 2 & 5 & 6 & 4 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 3

$$\begin{bmatrix} \text{KerD} & 4 & 13 & 18 & 10 & 4 & 1 \\ \text{Bett} & 0 & 3 & 6 & 6 & 4 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 4

$$\begin{bmatrix} \text{KerD} & 4 & 13 & 17 & 10 & 4 & 1 \\ \text{Bett} & 0 & 2 & 5 & 6 & 4 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 5

$$\begin{bmatrix} \text{KerD} & 4 & 13 & 18 & 10 & 5 & 1 \\ \text{Bett} & 0 & 3 & 6 & 7 & 5 & 1 \end{bmatrix}$$

Weight 2 of Engel ourType 6

$$\begin{bmatrix} \text{KerD} & 4 & 14 & 16 & 10 & 5 & 1 \\ \text{Bett} & 1 & 2 & 4 & 7 & 5 & 1 \end{bmatrix}$$

Weight 3, Common Headers

$$\begin{bmatrix} \text{m} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \text{SpaD} & 6 & 28 & 53 & 52 & 28 & 8 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 1

$$\begin{bmatrix} \text{KerD} & 6 & 26 & 33 & 28 & 16 & 6 & 1 \\ \text{Bett} & 4 & 6 & 9 & 16 & 14 & 6 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 2

$$\begin{bmatrix} \text{KerD} & 6 & 25 & 28 & 25 & 13 & 5 & 1 \\ \text{Bett} & 3 & 0 & 1 & 10 & 10 & 5 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 3

$$\begin{bmatrix} \text{KerD} & 6 & 25 & 30 & 25 & 13 & 5 & 1 \\ \text{Bett} & 3 & 2 & 3 & 10 & 10 & 5 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 4

$$\begin{bmatrix} \text{KerD} & 6 & 25 & 29 & 25 & 13 & 5 & 1 \\ \text{Bett} & 3 & 1 & 2 & 10 & 10 & 5 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 5

$$\begin{bmatrix} \text{KerD} & 6 & 25 & 30 & 25 & 16 & 5 & 1 \\ \text{Bett} & 3 & 2 & 3 & 13 & 13 & 5 & 1 \end{bmatrix}$$

Weight 3 of Engel ourType 6

$$\begin{bmatrix} \text{KerD} & 6 & 25 & 29 & 25 & 16 & 5 & 1 \\ \text{Bett} & 3 & 1 & 2 & 13 & 13 & 5 & 1 \end{bmatrix}$$

Table 3: by ZeroPlus-Engel-v3.mpl

5 Characteristic foliations

There is a notion of characteristic foliations in Engel theory, which is rank 1 distribution $\mathfrak{L}$ of Engel distribution D satisfying $[\mathfrak{L}, D^2] \subset D^2$ where $D^2 := D + [D,D]$ (, and $D^3 := D^2 + [D^2, D^2]$). In our six cases, they are given by $ay_1 + by_2$ for `mySol[3]`, and $C_{2,3,4}y_1 - y_2$ for the others, up to constant. It will be interesting to study their contributions to super homology discussion.

6 Another approach of finding Engel-like structures

So far, we studied Engel-like structures first fix dimensional conditions and next check Lie algebra structure. As [4] points out, there is a complete classification list of 4-dimensional Lie algebra by [10]. Since we unfortunately do not have access to the original paper [10], we use the list in [4].

$$\begin{aligned}
&\text{Type}[1] = \{[y_2, y_4] = y_1, [y_3, y_4] = y_2\}; \quad \text{Type}[2] = \{[y_1, y_4] = ay_1, [y_2, y_4] = y_2, [y_3, y_4] = y_2 + y_3\};\\
&\text{Type}[3] = \{[y_1, y_4] = y_1, [y_3, y_4] = y_2\}; \quad \text{Type}[4] = \{[y_1, y_4] = y_1, [y_2, y_4] = y_1 + y_2, [y_3, y_4] = y_2 + y_3\};\\
&\text{Type}[5] = \{[y_1, y_4] = y_1, [y_2, y_4] = ay_2, [y_3, y_4] = by_3, \text{ where } ab \neq 0\};\\
&\text{Type}[6] = \{[y_1, y_4] = ay_1, [y_2, y_4] = by_2 - y_3, [y_3, y_4] = y_2 + by_3, \text{ where } a \neq 0 \text{ and } b \geq 0\};\\
&\text{Type}[7] = \{[y_1, y_4] = 2y_1, [y_2, y_3] = y_1, [y_2, y_4] = y_2, [y_3, y_4] = y_2 + y_3\};\\
&\text{Type}[8] = \{[y_2, y_3] = y_1, [y_2, y_4] = y_2, [y_3, y_4] = -y_3\};\\
&\text{Type}[9] = \{[y_1, y_4] = (1+b)y_1, [y_2, y_3] = y_1, [y_2, y_4] = y_2, [y_3, y_4] = by_3, \text{ where } -1 < b \leqq 1\};\\
&\text{Type}[10] = \{[y_2, y_3] = y_1, [y_2, y_4] = -y_3, [y_3, y_4] = y_2\};\\
&\text{Type}[11] = \{[y_1, y_4] = 2ay_1, [y_2, y_3] = y_1, [y_2, y_4] = ay_2 - y_3, [y_3, y_4] = y_2 + ay_3\};
\end{aligned}$$

Type$[12] = \{[y_1, y_3] = y_1, [y_1, y_4] = -y_2, [y_2, y_3] = y_2, [y_2, y_4] = y_1\}$;

In this section, we first fix each 4-dimensional Lie algebra and try to find 2-dimensional subspace D satisfying Engel-like structure. We prepare a small proposition.

Proposition 6.1 Let $\mathfrak{g}$ be a 4-dimensional Lie algebra with bracket relations by a basis $(y_1, \ldots, y_4)$.
Take a 2-dimensional plane D where $w_1 = \sum_{i=1}^4 p_i y_i$ and $w_2 = \sum_{i=1}^4 q_i y_i$ are its basis. Put $w_3 = [w_1, w_2]$ and $w_4 = [w_1, w_3]$.
If $\{w_1, w_2, w_3\}$ and $\{w_1, w_2, w_3, w_4\}$ are linearly independent, i.e., $w_1 \wedge w_2 \wedge w_3 \neq 0$ and $w_1 \wedge w_2 \wedge w_3 \wedge w_4 \neq 0$ for some $\{p_i\}_{i=1}^4, \{q_i\}_{i=1}^4$, then $(\mathfrak{g}, D)$ is an Engel-like structure.

In this note, we call the scalar $(w_1 \wedge w_2 \wedge w_3 \wedge w_4)/(y_1 \wedge y_2 \wedge y_3 \wedge y_4)$ above by the *E-l-C* (Engel-like coefficient). Using the notation $\mathrm{Det}(i, j) = p_i q_j - p_j q_i = \begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix}$, the Engel-like coefficient of Lie algebras Type[i] (i=1..12) are given as follows.

E-l-C of Type$[1] = p_4 \mathrm{Det}(3,4)^3$

E-l-C of Type$[2] = (a-1)^2 p_4 \mathrm{Det}(1,4)\mathrm{Det}(3,4)^2$

E-l-C of Type$[3] = p_4 \mathrm{Det}(1,4)\mathrm{Det}(3,4)^2$

E-l-C of Type$[4] = p_4 \mathrm{Det}(3,4)^3$

E-l-C of Type$[5] = (a-1)(b-1)(a-b) p_4 \mathrm{Det}(1,4)\mathrm{Det}(2,4)\mathrm{Det}(3,4)$

E-l-C of Type$[6] = ((a-b)^2+1) p_4 \mathrm{Det}(1,4)(\mathrm{Det}(2,4)^2 + \mathrm{Det}(3,4)^2)$

E-l-C of Type$[7] = \mathrm{Det}(3,4)^2(p_4\mathrm{Det}(1,4) + p_4\mathrm{Det}(2,3) + p_3\mathrm{Det}(3,4))$

E-l-C of Type$[8] = -2\mathrm{Det}(2,4)\mathrm{Det}(3,4)(p_4\mathrm{Det}(1,4) - p_3\mathrm{Det}(2,4) - p_2\mathrm{Det}(3,4))$

E-l-C of Type$[9] = -(b-1)\mathrm{Det}(2,4)\mathrm{Det}(3,4)\left(p_3\mathrm{Det}(1,4) + b(p_4\mathrm{Det}(1,4) - p_2\mathrm{Det}(3,4))\right)$

E-l-C ofType$[10] = (\mathrm{Det}(2,4)^2 + \mathrm{Det}(3,4)^2)(p_4\mathrm{Det}(1,4) + p_2\mathrm{Det}(2,4) + p_3\mathrm{Det}(3,4))$

E-l-C of Type$[11] = (\mathrm{Det}(2,4)^2 + \mathrm{Det}(3,4)^2)(a^2 p_4\mathrm{Det}(1,4) + a p_4\mathrm{Det}(2,3) + p_4\mathrm{Det}(1,4) + p_2\mathrm{Det}(2,4) + p_3\mathrm{Det}(3,4$

E-l-C of Type$[12] = p_4\mathrm{Det}(3,4)\left(\mathrm{Det}(1,3)^2 + \mathrm{Det}(1,4)^2 + \mathrm{Det}(2,3)^2 + \mathrm{Det}(2,4)^2 + 2\mathrm{Det}(1,2)\mathrm{Det}(3,4)\right)$

In the case of Type[1], taking $p = [0,0,0,1]$ and $q = [0,0,1,0]$, i.e., $w_1 = y_4$ and $w_2 = y_3$ give an Engel-like structure.
In the case of Type[2], if $a = 1$, then there is no Engel-like structure. Assume $a \neq 1$, then $p = [0,0,0,1]$ and $q = [1,0,1,0]$ give an Engel-like structure.
In the case of Type[3], $p = [0,0,0,1]$ and $q = [1,0,1,0]$ give an Engel-like structure like as the second half of Type[2].
The case Type[4] is the same with Type[1].
In the case of Type[5], if $(a-1)(b-1)(a-b) = 0$, then there is no Engel-like structure. Otherwise, $p = [0,0,0,1]$ and $q = [1,0,1,0]$ give an Engel-like structure.
In the case of Type[6], $p = [0,0,0,1]$ and $q = [1,1,1,0]$ give an Engel-like structure.
In the case of Type[7], $p = [0,0,1,1]$ and $q = [0,0,0,1]$ give an Engel-like structure.
In the case of Type[8], $p = [0,0,0,1]$ and $q = [1,1,1,0]$ give an Engel-like structure.
In the case of Type[9], if $b - 1 = 0$ then there is no Engel-like structure. Otherwise, $p = [1,1,1,1]$ and $q = [0,0,0,1]$ give an Engel-like structure.
In the case of Type[10], $p = [0,0,0,1]$ and $q = [1,0,1,0]$ give an Engel-like structure.

In the case of Type[11], $p = [0,0,1,0]$ and $q = [0,0,0,1]$ give an Engel-like structure.
In the case of Type[12], $p = [0,1,0,1]$ and $q = [0,1,1,0]$ give an Engel-like structure.

References

[1] Jiro Adachi. Integral curves for contact and Engel structures. In *Real and complex singularities*, pages 1–32. World Sci. Publ., Hackensack, NJ, 2007.

[2] Jiro Adachi. Relative local equivalence of Engel structures. *Math. Z.*, 273(3-4):711–733, 2013.

[3] Rory Biggs and Claudiu C. Remsing. On the classification of real four-dimensional Lie groups. *J. Lie Theory*, 26(4):1001–1035, 2016.

[4] R. Ghanam and G. Thompson. Minimal matrix representations of four-dimensional Lie algebras. *Bull. Malays. Math. Sci. Soc. (2)*, 36(2):343–349, 2013.

[5] Kentaro Mikami and Tadayoshi Mizutani. Euler number and Betti numbers of homology groups of pre Lie superalgebra. arXiv:1809.08028v2, December 2018.

[6] Kentaro Mikami and Tadayoshi Mizutani. Euler number of homology groups of super Lie algebra. arXiv:1809.08028v1, September 2018.

[7] Kentaro Mikami and Tadayoshi Mizutani. Deformed super brackets on forms of a manifold. arXiv:2112.05896, Dec 2021.

[8] Kentaro Mikami and Tadayoshi Mizutani. Superalgebra structure on differential forms of manifold. arXiv:2105.09738, May 2021.

[9] Kentaro Mikami and Tadayoshi Mizutani. The second Betti number of doubly weighted homology groups of some pre Lie superalgebra. *Tohoku Math. J.*, 74(2):301–311, 2022. arXiv:1902.09137.

[10] G. M. Mubarakzjanov. Classification of real structures of Lie algebras of fifth order. *Izv. Vysš. Učebn. Zaved. Matematika*, 1963(3 (34)):99–106, 1963.

[11] Libor Šnobl and Pavel Winternitz. *Classification and identification of Lie algebras*, volume 33 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2014.

A Table of bracket for Engel-like Lie algebras

ourSolDep[1]

$$[y_1, y_4] = C_{1,4,3}y_3 + C_{1,4,4}y_4 \qquad [y_2, y_3] = (-C_{1,4,4}C_{2,3,4} + C_{2,4,4})y_3 + C_{2,3,4}y_4$$
$$[y_2, y_4] = (C_{1,4,3}C_{2,3,4})y_3 + C_{2,4,4}y_4 \qquad [y_3, y_4] = 0$$

ourSolDep[2]

$$[y_1, y_4] = \frac{C_{1,4,4}^2 + 4C_{1,4,3}}{8}\left(-(C_{1,4,4}C_{2,3,4} - 2C_{2,4,4})y_1 - C_{1,4,4}y_2\right) + C_{1,4,3}y_3 + C_{1,4,4}y_4$$

$$[y_2,y_3] = \frac{(C_{1,4,4}^2+4C_{1,4,3})(C_{1,4,4}C_{2,3,4}-C_{2,4,4})}{2C_{1,4,4}^2}(-(C_{1,4,4}C_{2,3,4}-2C_{2,4,4})y_1 - C_{1,4,4}y_2)$$
$$+(-C_{1,4,4}C_{2,3,4}+C_{2,4,4})y_3 + C_{2,3,4}y_4$$
$$[y_2,y_4] = -\frac{1}{8}(C_{1,4,4}C_{2,3,4}-2C_{2,4,4})(C_{1,4,4}^2+4C_{1,4,3})C_{2,3,4}y_1 - \frac{1}{8}(C_{1,4,4}^2+4C_{1,4,3})C_{1,4,4}C_{2,3,4}y_2$$
$$-\frac{1}{2C_{1,4,4}}(C_{1,4,4}^3C_{2,3,4}+2C_{1,4,3}C_{1,4,4}C_{2,3,4}-C_{1,4,4}^2C_{2,4,4}-4C_{1,4,3}C_{2,4,4})y_3 + C_{2,4,4}y_4$$
$$[y_3,y_4] = \frac{(C_{1,4,4}^2+4C_{1,4,3})(C_{1,4,4}C_{2,3,4}-C_{2,4,4})}{8C_{1,4,4}^2}\Big((C_{1,4,4}C_{2,3,4}-C_{2,4,4})(C_{1,4,4}^2+4C_{1,4,3})y_1$$
$$+C_{1,4,4}(C_{1,4,4}^2+4C_{1,4,3})y_2 + 2C_{1,4,4}^2y_3 + 4C_{1,4,4}y_4\Big)$$

ourSolDep[3]

$$[y_1,y_4] = -\frac{C_{1,4,2}C_{2,4,4}}{C_{1,4,4}}y_1 + C_{1,4,2}y_2 + C_{1,4,3}y_3 + C_{1,4,4}y_4 \qquad [y_2,y_3] = \frac{C_{2,4,4}}{C_{1,4,4}}y_4$$
$$[y_2,y_4] = \frac{C_{2,4,4}}{C_{1,4,4}^2}\Big(-C_{1,4,2}C_{2,4,4}y_1 + C_{1,4,2}C_{1,4,4}y_2 + C_{1,4,3}C_{1,4,4}y_3 + C_{1,4,4}^2y_4\Big) \qquad [y_3,y_4] = 0$$

ourSolDep[4]

$$[y_1,y_4] = 0 \qquad [y_2,y_3] = C_{2,3,1}y_1 + C_{2,4,4}y_3 + C_{2,3,4}y_4$$
$$[y_2,y_4] = C_{2,4,4}y_4 \qquad [y_3,y_4] = 0$$

ourSolDep[5]

$$[y_1,y_4] = (-C_{1,4,2}C_{2,3,4})y_1 + C_{1,4,2}y_2 + C_{1,4,3}y_3 \qquad [y_2,y_3] = C_{2,3,4}y_4$$
$$[y_2,y_4] = (-C_{1,4,2}C_{2,3,4}^2)y_1 + (C_{1,4,2}C_{2,3,4})y_2 + (C_{1,4,3}C_{2,3,4})y_3 \qquad [y_3,y_4] = 0$$

ourSolDep[6]

$$[y_1,y_4] = C_{1,4,3}y_3 \qquad [y_2,y_3] = C_{2,3,1}y_1 + C_{3,4,4}y_2 + C_{2,3,4}y_4$$
$$[y_2,y_4] = (C_{1,4,3}C_{2,3,4}+C_{3,4,4})y_3 \qquad [y_3,y_4] = (-C_{1,4,3}C_{2,3,1})y_1 + (-C_{1,4,3}C_{3,4,4})y_2 + C_{3,4,4}y_4$$

B About main maple scripts in this note

Only Engel-try-1.mpl had a 2 pages full explanation, but Engel-mySols-v1.mpl, DualAlg-Engel-v44.mpl, and ZeroPlus-Engel-v2.mpl had no chance to be explaned at all.
In stead, we invite readers to `\protect\vrule width0pt\protect\href{http://math.akita-u.ac.jp/\strin`
There we prepare a tar-ball which includes revised maple scripts with the extension ".mpl.with", a small repository `big.mla`, and `please-read-me.txt`.