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Euler numbers and Betti numbers of homology groups of pre Lie superalgebra

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As an analogy of Lie algebra homology, we study homology theory of Lie superalgebras. Specifically, we handle pre Lie superalgebra of multivector fields on Euclidean space with polynomial coefficients whose super Lie bracket is given by the Schouten bracket. In order to apply combinatorics, we introduce *weights*, especially *double weight* (w, h). Main result is that the Euler numbers are always equal to zero for each (w, h) and each Betti number is zero provided that $w \neq h$.

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1. Introduction

Well known de Rham cohomology group of a differentiable manifold M is a cohomology group of the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields on M together with the $\mathfrak{X}(M)$-module $C^\infty(M)$ as coefficient. Similarly, the Gel'fand-Fuks cohomology theory is a cohomology theory of infinite dimensional Lie algebras. There are many works on the cohomology of related subject, for example, the Lie algebra of volume preserving vector fields, the Lie algebra of formal Hamiltonian vector fields. The notion of these Lie algebra (co)homology groups is easy to understand, but the calculations are hard to complete and one of the reason is the infinity of dimensions. In order to reduce our computation to finite dimensional case, we use an idea of "weight" (c.f. for instance, [5], [4], [3], [2]).

There are (co)homology theories of Lie superalgebras but few works of $\mathbb{Z}$-graded version. Among Poisson geometry researchers, $\sum \Lambda^p \mathrm{T}(M)$ with the Schouten bracket is known as a prototype of $\mathbb{Z}$-graded (pre) Lie superalgebra and it is well-known that a 2-vector field π is Poisson if and only if $[\pi, \pi] = 0$. This Poisson condition $[\pi, \pi] = 0$ is equivalent to that $\partial(\pi \wedge \pi) = 0$ in superalgebra homology theory, and $\sqrt{\ker(\partial)}$ (the square root of cycles) is the space of Poisson structures in a formal sense and it seems there is some possibility of studying Poisson structures in this direction.

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Thus, in this note, we begin to study homology groups of pre Lie superalgebra including the second homology. Since in several works of Lie algebra (co)homology theory, the notion of *weight* plays important role, we will also introduce a notion of doubly weighted pre Lie superalgebras in an appropriate situation.

First we recall the definition of Lie superalgebra and pre Lie superalgebra.

Definition 1.1 ((pre) Lie superalgebra). Suppose a real vector space $\mathfrak{g}$ is graded by $\mathbb{Z}$ as $\mathfrak{g} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j$ and has a bilinear operation $[\cdot,\cdot]$ satisfying

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \tag{1.1}$$

$$[X,Y] + (-1)^{xy}[Y,X] = 0 \quad \text{where } X \in \mathfrak{g}_x \text{ and } Y \in \mathfrak{g}_y \tag{1.2}$$

$$(-1)^{xz}[[X,Y],Z] + (-1)^{yx}[[Y,Z],X] + (-1)^{zy}[[Z,X],Y] = 0\ . \tag{1.3}$$

Then we call $\mathfrak{g}$ a *pre (or $\mathbb{Z}$-graded) Lie superalgebra*.

A Lie *superalgebra* $\mathfrak{g}$ is graded by $\mathbb{Z}_2$ as $\mathfrak{g} = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ and the condition (1.1) is regarded as $[\mathfrak{g}_{[1]}, \mathfrak{g}_{[1]}] \subset \mathfrak{g}_{[0]}$ in modulo 2 sense.

Remark 1.1. Super Jacobi identity (1.3) above is equivalent to the one of the following.

$$[[X,Y],Z] = [X,[Y,Z]] + (-1)^{yz}[[X,Z],Y] \tag{1.4}$$

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{xy}[Y,[X,Z]] \tag{1.5}$$

Suppose $\mathfrak{g} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j$ is a pre Lie superalgebra. Let $\mathfrak{g}_{[0]} = \sum_{i \text{ is even}} \mathfrak{g}_i$ and $\mathfrak{g}_{[1]} = \sum_{i \text{ is odd}} \mathfrak{g}_i$. Then $\mathfrak{g} = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ holds and this is a Lie superalgebra.

Example 1.1. Take an n-dimensional vector space V and split it as $V = V_0 \oplus V_1$. Define $\mathfrak{g}_{[i]} = \{A \in \mathfrak{gl}(V) \mid A(V_j) \subset V_{i+j}\}$, where $\mathfrak{gl}(V)$ is the space of endomorphisms of V. For each $A \in \mathfrak{g}_{[i]}$ and $B \in \mathfrak{g}_{[j]}$, define $[A,B] = AB - (-1)^{ij}BA$. Then $\mathfrak{gl}(V) = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$ with this bracket is a Lie superalgebra.

More concretely, we take $n = 2$ and $\dim V_{[i]} = 1$ for $i = 0, 1$. Then $\mathfrak{g}_{[0]} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ and $\mathfrak{g}_{[1]} = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$. If we define $\mathfrak{g}_0 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$, $\mathfrak{g}_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $\mathfrak{g}_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Then $\mathfrak{gl}(2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a pre Lie superalgebra.

We will introduce the notion of *double weight* in pre Lie superalgebras (cf. Definition 2.4) and our results in this note are the calculation of the Euler number and Betti numbers of homology groups of double-weighted pre Lie superalgebras of special type.

- For general n, the Euler number of chain complex $\{\mathrm{C}_{\bullet,0,h}\}$ is 0 for each h. (Lemma 4.1)
- For general n, the Euler number of chain complex $\{\mathrm{C}_{\bullet,w,h}\}$ is 0 for each w and each h. (Theorem 4.1)
- The Euler number of $(\overline{\mathrm{C}}_{\bullet,w,h}, \partial_V)$ is 0 for each w and h. (Theorem 4.2)

- The m-th Betti number of $\{\mathrm{C}_{\bullet,w,h}\}$ is 0 for each double weight (w,h) if $w \neq h$. (Theorem 5.1)
- The first Betti number of $\{\mathrm{C}_{\bullet,w,h}\}$ is 0 for each double weight (w,h). (Theorem 5.2)

2. Preliminaries, Notations and Basic Facts

In a usual Lie algebra homology theory, m-th chain space is the exterior product $\Lambda^m\mathfrak{g}$ of $\mathfrak{g}$ and the boundary operator essentially comes from the operation $X \wedge Y \mapsto [X,Y]$.

Likewise, in the case of pre Lie superalgebras, "exterior algebra" is defined as the quotient of the tensor algebra $\otimes^m\mathfrak{g}$ of $\mathfrak{g}$ by the two-sided ideal generated by

$$X \otimes Y + (-1)^{xy} Y \otimes X \quad \text{where} \quad X \in \mathfrak{g}_x, Y \in \mathfrak{g}_y\ , \tag{2.1}$$

and we denote the equivalence class of $X \otimes Y$ by $X\Delta Y$. Since $X_{\text{odd}}\Delta Y_{\text{odd}} = Y_{\text{odd}}\Delta X_{\text{odd}}$ and $X_{\text{even}}\Delta Y_{\text{any}} = -Y_{\text{any}}\Delta X_{\text{even}}$ hold, $\Delta^m\mathfrak{g}_k$ has a symmetric property for odd k and has a skew-symmetric property for even k with respect to Δ.

Definition 2.1. Assume that the pre Lie superalgebra $\mathfrak{g}$ acts on a module V as follows: For each homogeneous $\xi \in \mathfrak{g}$ there corresponds an element $\xi_V \in \mathrm{End}(V)$ and satisfies $[\xi,\eta]_V = \xi_V \circ \eta_V - (-1)^{|\xi||\eta|}\eta_V \circ \xi_V$ where $\xi \in \mathfrak{g}_{|\xi|}$ and $\eta \in \mathfrak{g}_{|\eta|}$. Then we call V a $\mathfrak{g}$-module. We often write $\xi \cdot v$ for $\xi_V(v)$.

Example 2.1. A pre Lie superalgebra $\mathfrak{g}$ is itself a $\mathfrak{g}$-module by own bracket $X \cdot Z = [X,Z]$. Let $X, Y \in \mathfrak{g}$ be homogeneous as $X \in \mathfrak{g}_x, Y \in \mathfrak{g}_y$. Then, $(X \circ Y - (-1)^{xy} Y \circ X) \cdot Z = [X,Y] \cdot Z$ holds and this is just Jacobi identity.

Suppose we have an exterior product $Y_1\Delta \cdots \Delta Y_m$ of $Y_1, \ldots, Y_m$. Omitting i-th element, we have $Y_1\Delta \cdots \Delta Y_{i-1}\Delta Y_{i+1}\Delta \cdots \Delta Y_m$. It is often denoted as $Y_1\Delta \cdots \widehat{Y_i} \cdots \Delta Y_m$. Here we denote it by $\widehat{\mathfrak{Y}_m}[i]$. If we omit i-th and j-th elements, we denote the omitted product by $\widehat{\mathfrak{Y}_m}[i,j]$.

Definition 2.2. Let V be a $\mathfrak{g}$-module. For integer $m > 0$, define $\overline{\mathrm{C}}_m = \Delta^m\mathfrak{g} \otimes V = \sum\limits_{i_1 \leq \ldots \leq i_m} \mathfrak{g}_{i_1}\Delta \cdots \Delta\mathfrak{g}_{i_m} \otimes V$, which is called m-th *chain space with coefficient in* V. In the case where $m = 0$, we define $\overline{\mathrm{C}}_0 = V$.

We define a linear map $\partial_V : \overline{\mathrm{C}}_m \to \overline{\mathrm{C}}_{m-1}$ called (*boundary homomorphism*) by

$$\begin{aligned}
&\partial_V(Y_1\Delta \cdots \Delta Y_m \otimes v) \\
&= \sum_{i<j} (-1)^{\sum_{s<j}(1+y_j y_s) + \sum_{s<i}(1+y_i y_s)} [Y_j, Y_i]\Delta\widehat{\mathfrak{Y}_m}[i,j] \otimes v \qquad (2.2) \\
&\quad + (-1)^{m+1} \sum_{i=1}^{m} (-1)^{\sum_{s>i}(1+y_i y_s)} \widehat{\mathfrak{Y}_m}[i] \otimes Y_i \cdot v \qquad (2.3)
\end{aligned}$$

for a decomposable element, where y_i is the degree of homogeneous element Y_i, i.e., $Y_i \in \mathfrak{g}_{y_i}$.

We have the following basic fact.

Theorem 2.1. $\partial_V \circ \partial_V = 0$ *holds and we have* m*-th homology group denoted by*

$$H_m(\mathfrak{g}, V) = \ker(\partial_V : \overline{C}_m \to \overline{C}_{m-1})/\partial_V(\overline{C}_{m+1}) \ .$$

Remark 2.1. The first term of (2.2) is also expressed as

$$\sum_{i<j}(-1)^{i-1+y_i \sum_{i<s<j} y_s} Y_1 \Delta \cdots \widehat{Y_i} \cdots \Delta \underbrace{[Y_i, Y_j]}_{j} \Delta \cdots \Delta Y_m \otimes v \ . \tag{2.4}$$

Remark 2.2. If $\mathfrak{g}$-action on V is trivial, namely $Y \cdot v = 0$ for $\forall Y \in \mathfrak{g}$ and $\forall v \in V$, then (2.3) is always 0 and we may assume $V = \mathbb{R}$. We call this module the trivial module. Thus, when we essentially deal with the trivial module, the chain space $\mathrm{C}_m = \Delta^m \mathfrak{g}$ and ∂_V is (2.2) without v, which we denote ∂. It is clear that $\partial \circ \partial = 0$ and we have the homology groups

$$\mathrm{H}_m(\mathfrak{g}, \mathbb{R}) = \ker(\partial : \mathrm{C}_m \to \mathrm{C}_{m-1})/\partial(\mathrm{C}_{m+1}) \ .$$

2.1. *Homology groups weighted by the first grading*

Assume that a $\mathfrak{g}$-module V is $\mathbb{Z}$-graded, i.e, $V = \sum_i V_i$, and the action satisfies $\mathfrak{g}_i \cdot V_j \subset V_{i+j}$.

Definition 2.3. We define a non-zero element in $\mathfrak{g}_{i_1} \Delta \cdots \Delta \mathfrak{g}_{i_m} \otimes V_j$ to have $i_1 + \cdots + i_m + j$ as the (first) weight. Define the subspace of $\overline{\mathrm{C}}_m$ by

$$\overline{\mathrm{C}}_{m,w} = \sum_{\substack{i_1 \leq \ldots \leq i_m \\ \sum_{s=1}^m i_s + j = w}} \mathfrak{g}_{i_1} \Delta \cdots \Delta \mathfrak{g}_{i_m} \otimes V_j \ ,$$

which is the direct sum of different types of spaces of elements with weight w.

Proposition 2.1. *The (first) weight* w *is preserved by* ∂_V*, i.e., we have* $\partial_V(\overline{C}_{m,w}) \subset \overline{C}_{m-1,w}$*. Thus, for a fixed* w*, we have* w*-weighted homology groups*

$$H_{m,w}(\mathfrak{g}, V) = \ker(\partial_V : \overline{C}_{m,w} \to \overline{C}_{m-1,w})/\partial(\overline{C}_{m+1,w}) \ .$$

When V *is the trivial module, we have*

$$H_{m,w}(\mathfrak{g}, \mathbb{R}) = \ker(\partial : C_{m,w} \to C_{m-1,w})/\partial(C_{m+1,w}) \ .$$

2.2. *Double-weighted homology groups*

Definition 2.4 (Double weight). Assume that each subspace $\mathfrak{g}_i$ of a given pre Lie superalgebra $\mathfrak{g}$ is directly decomposed into subspaces $\mathfrak{g}_{i,j}$ as $\mathfrak{g}_i = \sum_j \mathfrak{g}_{i,j}$ and satisfies

$$[X, Y] \in \mathfrak{g}_{i_1+i_2, j_1+j_2} \quad \text{for each } X \in \mathfrak{g}_{i_1,j_1} \ , \ Y \in \mathfrak{g}_{i_2,j_2} \ . \tag{2.5}$$

We say such a pre Lie superalgebra is *double-weighted*.

Assume that $\mathfrak{g}$-module V is also double-weighted $V_{i,j}$ and satisfies $\mathfrak{g}_{i,j} \cdot V_{i',j'} \subset V_{i+i',j+j'}$.

Then we define *double-weighted m-th chain space* by

$$\overline{\mathrm{C}}_{m,w,h} = \sum_{\substack{i_1 \leq \ldots \leq i_m \ , \ \sum_{s=1}^m i_s + i_0 = w \\ \sum_{s=1}^m h_s + h_0 = h}} \mathfrak{g}_{i_1,h_1} \Delta \cdots \Delta \mathfrak{g}_{i_m,h_m} \otimes V_{i_0,h_0}$$

Proposition 2.2. *The double weight (w, h) is preserved by ∂_V, i.e., we have $\partial_V(\overline{C}_{m,w,h}) \subset \overline{C}_{m-1,w,h}$. Thus, we have (w, h)-weighted homology groups*

$$H_{m,w,h}(\mathfrak{g}, V) = \ker(\partial_V : \overline{C}_{m,w,h} \to \overline{C}_{m-1,w,h})/\partial(\overline{C}_{m+1,w,h}) \ .$$

When V is the trivial module, then we have

$$H_{m,w,h}(\mathfrak{g}, \mathbb{R}) = \ker(\partial : C_{m,w,h} \to C_{m-1,w,h})/\partial(C_{m+1,w,h}) \ .$$

3. Pre Lie Superalgebras with the Schouten Bracket

A prototype of pre Lie superalgebra is the exterior algebra of the sections of exterior power of tangent bundle of a differentiable manifold M. We denote the i-th exterior bundle by $\Lambda^i \mathrm{T}(M)$ and for the sake of simplicity we express its space of sections by the same notation. Let $n = \dim M$ and put

$$\mathfrak{g} = \sum_{i=1}^{n} \Lambda^i \mathrm{T}(M) = \sum_{i=0}^{n-1} \mathfrak{g}_i \ , \quad \text{where} \quad \mathfrak{g}_i = \Lambda^{i+1}\mathrm{T}(M)$$

with the Schouten bracket.

There are several ways of defining the Schouten bracket, namely, axiomatic explanation, sophisticated one using Clifford algebra or more direct ones (cf. [6]). Here in the context of Lie algebra homology theory, we introduce the Schouten bracket as follows:

Definition 3.1 (Schouten bracket). Let ∂ denote the boundary operator for the Lie algebra of vector fields on M. For $A \in \Lambda^a \mathrm{T}(M)$ and $B \in \Lambda^b \mathrm{T}(M)$, we define a binary operation $[\cdot, \cdot]$ by the following and call $[A, B]$ the *Schouten bracket* of A and B.

$$(-1)^{a+1}[A, B] = \partial(A \wedge B) - (\partial A) \wedge B - (-1)^a A \wedge \partial B \ . \tag{3.1}$$

Thus the Schouten bracket measures how far from the derivation the boundary operator ∂ is.

The first chain space is $\mathrm{C}_1 = \mathfrak{g} = \sum_{p=1}^{n} \Lambda^p \mathrm{T}(M)$. The second chain space is given by

$$\begin{aligned} \mathrm{C}_2 = \mathfrak{g}\Delta\mathfrak{g} =& \sum_{p \leq q} \Lambda^p \mathrm{T}(M) \Delta \Lambda^q \mathrm{T}(M) \\ =& \Lambda^1 \mathrm{T}(M) \Delta \Lambda^1 \mathrm{T}(M) + \Lambda^1 \mathrm{T}(M) \Delta \Lambda^2 \mathrm{T}(M) + \cdots \\ &+ \Lambda^2 \mathrm{T}(M) \Delta \Lambda^2 \mathrm{T}(M) + \Lambda^2 \mathrm{T}(M) \Delta \Lambda^3 \mathrm{T}(M) + \cdots \ . \end{aligned}$$

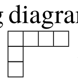
Remark 3.1. Let $\pi \in \Lambda^2\mathrm{T}(M)$. Then $\pi\Delta\pi \in \Lambda^2\mathrm{T}(M)\Delta\Lambda^2\mathrm{T}(M) \subset \mathrm{C}_2$ and $\partial(\pi\Delta\pi) = [\pi,\pi] \in \mathrm{C}_1$. Thus, $\pi \in \Lambda^2\mathrm{T}(M)$ is Poisson if and only if $\partial(\pi\Delta\pi) = 0$, and we express it by $\pi \in \sqrt{\ker(\partial)}$ symbolically. It will be interesting to study $\sqrt{\ker(\partial)}$ and also interesting to study specific properties of Poisson structures in $\sqrt{\partial(\mathrm{C}_3)}$, which come from the boundary image of the third chain space C_3.

In this pre Lie superalgebra, possible weights are non-negative integers. When weight is 0, the chain spaces with trivial action are simply given by $\mathrm{C}_{m,0} = \Delta^m\mathfrak{g}_0 = \Delta^m\mathrm{T}(M)$ and the homology is the Lie algebra homology of vector fields for $m = 1, \ldots, n$. For lower weights 1 or 2, the chain spaces are given by

$$\begin{aligned}
\mathrm{C}_{m,1} &= \Delta^{m-1}\mathfrak{g}_0\Delta\mathfrak{g}_1 = \Delta^{m-1}\mathrm{T}(M)\Delta\Lambda^2\mathrm{T}(M) \quad \text{for} \quad m = 1, \ldots\ , \\
\mathrm{C}_{m,2} &= \Delta^{m-1}\mathfrak{g}_0\Delta\mathfrak{g}_2 \oplus \Delta^{m-2}\mathfrak{g}_0\Delta^2\mathfrak{g}_1 \\
&= \Delta^{m-1}\mathrm{T}(M)\Delta\Lambda^3\mathrm{T}(M) \oplus \Delta^{m-2}\mathrm{T}(M)\Delta^2\Lambda^2\mathrm{T}(M) \quad \text{for} \quad m = 1, \ldots\ .
\end{aligned}$$

Remark 3.2. In particular, $\mathrm{C}_{1,2} = \Lambda^3\mathrm{T}(M)$, $\mathrm{C}_{2,2} = \mathrm{T}(M)\Delta\Lambda^3\mathrm{T}(M) \oplus \Lambda^2\mathrm{T}(M)\Delta\Lambda^2\mathrm{T}(M)$, $\mathrm{C}_{3,2} = \mathrm{T}(M)\Delta\mathrm{T}(M)\Delta\Lambda^3\mathrm{T}(M) \oplus \mathrm{T}(M)\Delta\Lambda^2\mathrm{T}(M)\Delta\Lambda^2\mathrm{T}(M)$. Thus, by introducing weight, the chain spaces become smaller and computations become a little clear and easier.

Given an integer w for a weight, and the sequence $0 \leq i_1 \leq \cdots \leq i_m \leq n-1$ with $\sum_{s=1}^m i_s = w$, putting $j_s = 1 + i_s$ one obtains a new sequence of positive integers satisfying $1 \leq j_1 \leq \cdots \leq j_m \leq n$ with $\sum_{s=1}^m j_s = m + w$. This latter sequence $j_m, \ldots, j_1$ describes a Young diagram of area $w + m$ and length m. From each Young diagram $\{j_m, \ldots, j_1\}$, looking at the 'multiplicity' j_i in $\Delta^{j_i}\Lambda^i T(M)$, we obtain a sequence $[k_1, k_2, \ldots, k_n]$ consisting of $k_i = \#\{s | j_s = i\}$.

Remark 3.3 (3 ways of Young diagram). A Young diagram λ is a non-decreasing sequence of positive integers, say $a_1, \ldots, a_m$. For instance,  is a sequence of $4, 1, 1$, here we denote it as ${}^t(4,1,1)$ where superscript t means "traditional expression". As explained above, when we focus on the multiplicity of elements, we have another sequence, in the present example above, $2, 0, 0, 1$ and we denote it by $[2, 0, 0, 1]$. Sometimes we have to write many 0 in this expression. The 3rd expression of Young diagram is measuring the height of each column from left to right. Again in the example, we have a sequence $3, 1, 1, 1$ and denote it by $\langle 3, 1, 1, 1\rangle$ and call it *tower (vertical) decomposition*. It is known in general that the sequence of tower decomposition of λ is just the conjugate of λ, i.e, $\langle\lambda\rangle = {}^t(\text{conjugate of } \lambda)$. In detail of relations of those, refer to [6].

Remark 3.4. We remark that m does not stop at $\dim M$ in general because of the property of our new "wedge product" $\triangle$.

4. Euler Number of Homology Groups of Concrete pre Lie Superalgebras

In the previous section, we defined pre Lie superalgebras $\displaystyle\sum_{i=1}^n \Lambda^i\mathrm{T}(M)$ for differentiable manifold M. In this section, we treat the Euclidean space $M = \mathbb{R}^n$ with the Cartesian coordinates $x_1, \ldots, x_n$. Then, we consider a pre Lie super subalgebra consisting of multi vector fields with polynomial coefficients. We define $\mathfrak{g}_{i,j}$ as follows.

$$\mathfrak{g}_{i,j} = \mathfrak{X}^{i+1}_{j+1}(\mathbb{R}^n) = \{(i+1)\text{-multi vector fields with } (j+1)\text{-homogeneous polynomials}\} .$$

We see easily that $[\mathfrak{g}_{i_1,j_1}, \mathfrak{g}_{i_2,j_2}] \subset \mathfrak{g}_{i_1+i_2,j_1+j_2}$ and we get a double-weighted pre Lie superalgebra. The spaces $\mathfrak{g}_{i,j}$ are finite dimensional, precisely $\dim \mathfrak{g}_{i,j} = \binom{n-1+j+1}{n-1}\binom{n}{i+1}$. In the following subsection, we study chain space of $\mathrm{C}_{m,w,h}$.

4.1. *Double-weighted chain space $C_{m,w,h}$*

The chain space $\mathrm{C}_{m,w,h} = \sum \mathfrak{X}^{i_1}_{h_1}(\mathbb{R}^n)\Delta \cdots \Delta\mathfrak{X}^{i_m}_{h_m}(\mathbb{R}^n)$ of the double-weighted pre Lie superalgebra has the following properties.

(1) $(i_s)_{s=1}^m$ are non-descending sequences of sum $w+m$ and length m. Since each entry i_s is less than $n+1$, we may count the multiplicity of them as $k_a = \#\{s \mid i_s = a\}$ or denote it by $a\!:\!k_a$, and get $[k_1, \ldots, k_n]$, i.e.,

$$(i_1, \ldots, i_m) = (\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}) = (1\!:\!k_1, \ldots, n\!:\!k_n) = [k_1, \ldots, k_n] ,$$

we have

$$\sum_{s=1}^{n} k_s = m , \text{ and } \sum_{s=1}^{n} sk_s = w + m .$$

Now denote $\mathfrak{X}^{i_1}_{h_1}(\mathbb{R}^n)\Delta \cdots \Delta\mathfrak{X}^{i_m}_{h_q}(\mathbb{R}^n)$ by $\mathfrak{X}^{(i_1 \leq \cdots \leq i_m)}_{(h_1,\ldots,h_m)} = \mathfrak{X}^{[k_1,\ldots,k_n]}_{(h_1,\ldots,h_m)}$.

(2) Each h_s is non-negative integer and $\sum_{s=1}^m (h_s - 1) = h$ holds. In order to apply an idea of Young diagrams of positive integers, we have to shift each h_s by 1 as $h_s + 1$.

Then the second weight condition implies $\sum_{s=1}^m (h_s + 1) = h + 2m$ and we may consider the Young diagrams of area $w + 2m$ and length m. To recover original sequences, we shift them back by -1 simultaneously. Obtained sequences from these Young diagrams are ordered ones, so to obtain all of the original sequences of weights, we need permutations of them.

(3) Assume $i_{p-1} < i_p = \cdots = i_q = k < i_{q+1}$. Then we may relabel so that $h_p \leq \cdots \leq h_q$, and we write

$$\mathfrak{X}^k_{h_p}(\mathbb{R}^n)\Delta \cdots \Delta\mathfrak{X}^k_{h_q}(\mathbb{R}^n) = \mathrm{SubC}^{(k:(q-p+1))}(h_p, \ldots, h_q)$$

(4) Assume $i_p = \cdots = i_q$ and $h_p = \cdots = h_q$. Then, we use the notation

$$\mathfrak{X}^{i_p}_{h_p}(\mathbb{R}^n)\Delta \cdots \Delta\mathfrak{X}^{i_q}_{h_q}(\mathbb{R}^n) = \mathrm{SubC}^{(i_p:(q-p+1))}(\underbrace{h_p, \ldots, h_p}_{q-p+1}) = \Delta^{q-p+1}\mathfrak{X}^{i_p}_{h_p}(\mathbb{R}^n)$$

- If i_p is even, then $\Delta^{q-p}\mathfrak{X}^{i_p}_{h_p}(\mathbb{R}^n)$ is a symmetric space and its dimension is

$$\binom{\binom{n}{i_p}\binom{n-1+h_p}{n-1} - 1 + q - p + 1}{q - p + 1}$$

- If i_p is odd, then $\Delta^{q-p+1}\mathfrak{X}^{i_p}_{h_p}(\mathbb{R}^n)$ is a skew-symmetric space and its dimension is
$$\binom{\binom{n}{i_p}\binom{n-1+h_p}{n-1}}{q-p+1}$$
In particular, if $q-p+1 > \binom{n}{i_p}\binom{n-1+h_p}{n-1}$ then the space is 0-dimensional.

Introducing a new notation
$$\mathrm{SubC}^{(k:\ell)}_{[u]} = \bigoplus_{\sum_{s=1}^{\ell}(h_s-1)=u} \mathrm{SubC}^{(k:\ell)}(h_1,\dots,h_\ell)\ , \tag{4.1}$$
and using the notations above we have

Proposition 4.1.
$$C_{m,w,h} = \sum_{\substack{\sum_{i=1}^n k_i=m \\ \sum_{i=1}^n ik_i=w+m \\ \sum_{i=1}^n u_i=h}} \mathrm{SubC}^{(1:k_1)}_{[u_1]}\,\Delta\cdots\Delta\,\mathrm{SubC}^{(n:k_n)}_{[u_n]} \tag{4.2}$$

Proposition 4.2. *Assume k is an odd integer. Let $[\ell_1,\ell_2,\dots]$ be the sequence of multiplicities of $h_1+1,\dots,h_m+1$, that is, $\ell_b = \#\{i \mid h_i+1=b\}$. Then*
$$\mathrm{SubC}^{(k:m)}(h_1,\dots,h_m) = \Delta^{\ell_1}\mathfrak{X}^k_0\Delta^{\ell_2}\mathfrak{X}^k_1\Delta\cdots$$
holds. If an inequality $\ell_i \le \dim\mathfrak{X}^k_{i-1} = \binom{n}{k}\binom{n-1+i-1}{n-1}$ holds for each i, then $\mathrm{SubC}^{(k:m)}(h_1,\dots,h_m)$ is non trivial, and whose dimension is $\prod_i \binom{\binom{n}{k}\binom{n-1+i-1}{n-1}}{\ell_i}$.

Proof. Since k is odd, each space $\Delta^{\ell_i}\mathfrak{X}^k_{i-1}$ is skew-symmetric and the proposition holds comparing the dimension of $\mathfrak{X}^k_{i-1}$. □

The requirements on the chain space $\mathrm{C}_{m,w,h}$ in (4.2) are $w=\sum_{s=1}^m (s-1)k_s$ and $\sum_{s=1}^m k_s = m$ for the first weight. Thus, if $w=0$, then $[k_1,k_2,\dots]=[m,0,\dots]$ or if $w=1$, then $[k_1,k_2,\dots]=[m-1,1,0,\dots]$. If $w=2$, then $[k_1,k_2,\dots]=[m-2,2,0,\dots]$ or $[m-1,0,1,0,\dots]$.

From the definition of the second weight h, we see that $\sum_{s=1}^m h_s = m+h$, and so $m\ge -h$, more precisely, $m \ge \max(-h,1)$. About the upper bound of the range of m, we discuss later.

4.1.1. *Case where the first weight $w=0$*

In this subsection, we assume $w=0$. Then the product by Δ is always skew-symmetric and we see that
$$\mathrm{C}_{m,0,h} = \mathrm{SubC}^{(1:m)}_{[h]} = \sum_{\substack{\sum_t \ell_t=m \\ \sum_t t\ell_t=2m+h}} \Delta^{\ell_1}\mathfrak{X}^1_0\Delta^{\ell_2}\mathfrak{X}^1_1\Delta\cdots\ .$$

We have some restrictions on h and the range of m from the proposition 4.2.

Proposition 4.3. *Assume $w = 0$ and $C_{m,0,h} \neq 0$.*
Then $h \geq -\dim \mathfrak{X}^1_0$, and $\max(1, -h) \leq m \leq h + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$ holds.

Proof. We follow the notation above, then

$$\sum_t \ell_t = m \tag{4.3}$$

$$\sum_t t\ell_t = 2m + h \tag{4.4}$$

Since (4.4) $-$ 2(4.3), we have $-\ell_1 + \sum_{s>2}(s-2)\ell_s = h$, and $\sum_{s>2}(s-2)\ell_s = h + \ell_1$, thus we have $0 \leq h + \ell_1$. Applying the requirement $\ell_1 \leq \dim \mathfrak{X}^1_0$, we have $0 \leq h + \dim \mathfrak{X}^1_0$.

From (4.4) $-$ 3(4.3), we have $-2\ell_1 - \ell_2 + \sum_{s>3}(s-3)\ell_s = -m + h$, thus $0 \leq \sum_{s>3}(s-3)\ell_s = -m + h + 2\ell_1 + \ell_2$. Applying the requirement $\ell_2 \leq \dim \mathfrak{X}^1_1$, we have $m - h \leq 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$. □

Remark 4.1. In the previous proposition, we have an upper bound of m. If we use the third or higher comparison, we have more sharp estimate of upper bound of m.

Example 4.1. Assuming $n = 2$, we study the chain complex of

$$\cdots \leftarrow \mathrm{C}_{m-1,0,h} \xleftarrow{\partial} \mathrm{C}_{m,0,h} \xleftarrow{\partial} \mathrm{C}_{m+1,0,h} \xleftarrow{\partial} \cdots$$

where

$$\mathrm{C}_{m,0,h} = \sum_{\substack{\sum_t \ell_t = m \\ \sum_t t\ell_t = 2m+h}} \Delta^{\ell_1}\mathfrak{X}^1_0 \Delta^{\ell_2}\mathfrak{X}^1_1 \Delta \cdots \quad ,$$

and denote $d_m = \dim \mathrm{C}_{m,0,h}$ the dimension of m-chain space, $r_m = \dim \partial(\mathrm{C}_{m+1,0,h})$ for the rank ($\dim \partial$ symbolically). The m-Betti number is defined by $d_m - (r_{m-1} + r_m)$.

Assume $h = -2$. Then m starts from 2. The possible Young diagrams are characterized by area $2m - 2$ and length m. We see that the Young diagram $\langle m, m-2 \rangle = [1^2, 2^{m-2}]$ is the only possible candidate for our chain space. Thus $\mathrm{C}_{m,0,-2} = \Delta^2\mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-2}\mathfrak{X}^1_1(\mathbb{R}^2)$ and we get dimension, rank and Betti number for each space as in the table 4.1 left: The Euler number is 0.

Assume $h = -1$. The area is $2m - 1$, and the good Young diagrams are $\langle m, m-1 \rangle$ or $\langle m, m-2, 1 \rangle$ and so $[1^1, 2^{m-1}]$ or $[1^2, 2^{m-3}, 3^1]$. Thus

$$\mathrm{C}_{m,0,-1} = \mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-1}\mathfrak{X}^1_1(\mathbb{R}^2) \oplus \Delta^2\mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-3}\mathfrak{X}^1_1(\mathbb{R}^2)\Delta\mathfrak{X}^1_2(\mathbb{R}^2)\ .$$

So we get dimension, rank and Betti number for each space as in the table 4.1 right: The Euler number is 0.

Assume $h = 0$. The area is $2m$ and good Young diagrams are $\langle m, m \rangle$, $\langle m, m-1, 1 \rangle$, $\langle m, m-2, 1, 1 \rangle$ or $\langle m, m-2, 2 \rangle$ and so $[1^0, 2^m]$, $[1^1, 2^{m-2}, 3^1]$, $[1^2, 2^{m-3}, 4^1]$ or $[1^2, 2^{m-4}, 3^2]$. Thus

$$\mathrm{C}_{m,0,0} = \Delta^m\mathfrak{X}^1_1(\mathbb{R}^2) \oplus \mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-2}\mathfrak{X}^1_1(\mathbb{R}^2)\Delta\mathfrak{X}^1_2(\mathbb{R}^2)$$

Table 1. Left is $w = 0$ and $h = -2$, Right is $w = 0$ and $h = -1$

m	2	3	4	5	6
dim	1	4	6	4	1
$\dim \partial$	1	3	3	1	0
Betti	0	0	0	0	0

m	1	2	3	4	5	6	7
dim	2	8	18	32	38	24	6
$\dim \partial$	2	6	12	20	18	6	0
Betti	0	0	0	0	0	0	0

$$\oplus\, \Delta^2 \mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-3}\mathfrak{X}^1_1(\mathbb{R}^2)\Delta\mathfrak{X}^1_3(\mathbb{R}^2) \oplus \Delta^2\mathfrak{X}^1_0(\mathbb{R}^2)\Delta^{m-4}\mathfrak{X}^1_1(\mathbb{R}^2)\Delta^2\mathfrak{X}^1_2(\mathbb{R}^2)\ .$$

When $h = 0$, zero-th chain space is defined and $\mathrm{C}_{0,0,0} = \mathbb{R}$. Thus, the dimension for each space and the rank of ∂ are as follows: the Euler number is 0. and we get dimension, rank and Betti number for each space as in the table below:

Table 2. $w = 0$ and $h = 0$

m	0	1	2	3	4	5	6	7	8
dim	1	4	18	60	120	156	134	68	15
$\dim \partial$	0	4	14	46	74	80	54	13	0
Betti	1	0	0	0	0	2	0	1	2

We discussed in [6] the Euler numbers of Lie algebra homology groups of given weight w and homogeneity h of Poisson tensor where we dealt Young diagrams of area $w+(2-h)m$ with length m. By discussion there, we have the following result.

Lemma 4.1. *For general n, the Euler number of chain complex $\{C_{\bullet,0,h}\}$ is 0.*

Proof. We use the notations in [6]. Since

$$\mathrm{C}_{m,0,h} = \sum_{\substack{\sum_t \ell_t = m \\ \sum_t t\ell_t = 2m+h}} \Delta^{\ell_1}\mathfrak{X}^1_0\Delta^{\ell_2}\mathfrak{X}^1_1\Delta\cdots \quad ,$$

we have to deal with Young diagrams ∇^{2m+h}_m of area $2m+h$ with length m. A recursive formula

$$\nabla^{2m+h}_m = B\cdot\nabla^{2m+h-1}_{m-1} \sqcup T_m\nabla^{m+h}_m = T_m\cdot T_m \sqcup B\cdot\nabla^{2m-1}_{m-1} \tag{4.5}$$

holds. If $h = 0$ then we have

$$\nabla^{2m}_m = T_m\cdot T_m \sqcup B\cdot\nabla^{2m-1}_{m-1}\ .$$

Thus, $\dim \mathrm{C}_{m,0,0} = \binom{\dim \mathfrak{X}^1_1}{m} + \sum_{\lambda\in\nabla^{2m-1}_{m-1}} \dim(B\cdot\lambda)$. $\sum_{m>0}(-1)^m\binom{\dim\mathfrak{X}^1_1}{m} = -1$ is well-known.

When we denote each $\lambda \in \nabla^{2m-1}_{m-1}$ by $[\ell_1, \ell_2, \dots]$, $\sum_t \ell_t = m-1$ and $\sum_t t\ell_t = 2m-1$ must be satisfied. $B\cdot\lambda = [1+\ell_1, \ell_2, \dots]$ and so $\dim(B\cdot\lambda) = \binom{\dim\mathfrak{X}^1_0}{1+\ell_1}\binom{\dim\mathfrak{X}^1_1}{\ell_2}\binom{\dim\mathfrak{X}^1_2}{\ell_3}\dots$.

About the alternating sum of the second term, we have

$$\sum_{m>0}(-1)^m \sum_{\lambda\in\nabla^{2m-1}_{m-1}} \dim(B\cdot\lambda)$$

$$
\begin{aligned}
&= \sum_{m>0} (-1)^m \sum_{\substack{\sum_t \ell_t = m-1 \\ \sum_t t\ell_t = 2m-1}} \binom{\dim \mathfrak{X}^1_0}{1+\ell_1}\binom{\dim \mathfrak{X}^1_1}{\ell_2}\binom{\dim \mathfrak{X}^1_2}{\ell_3}\dots \\
&= \sum (-1)^{1+\sum_t \ell_t} \sum_{2(1+\sum_t \ell_t)=1+\sum_t t\ell_t} \binom{\dim \mathfrak{X}^1_0}{1+\ell_1}\binom{\dim \mathfrak{X}^1_1}{\ell_2}\binom{\dim \mathfrak{X}^1_2}{\ell_3}\dots \\
&= \sum (-1)^{\ell_2}\binom{\dim \mathfrak{X}^1_1}{\ell_2} \sum_{2(1+\sum_t \ell_t)=1+\sum_t t\ell_t} (-1)^{1+\sum_{t\neq 2}\ell_t}\binom{\dim \mathfrak{X}^1_0}{1+\ell_1}\binom{\dim \mathfrak{X}^1_2}{\ell_3}\dots \\
&= 0 \quad \text{because } \ell_2 \text{ is free in the condition } 2(1+\sum_t \ell_t) = 1 + \sum_t t\ell_t.
\end{aligned}
$$

So, $\sum_{m>0}(-1)^m \mathrm{C}_{m,0,0} = \sum_{m>0}(-1)^m\binom{\dim \mathfrak{X}^1_1}{m} = -1$. When $h=0$, then 0-th chain space $\mathrm{C}_{0,0,0}$ is defined and trivially 1-dimensional. Thus, the Euler number $\sum_{m\geq 0}(-1)^m \mathrm{C}_{m,0,0} = 0$.

When $h<0$ then (4.5) says that $\nabla^{2m+h}_m = B\cdot\nabla^{2m+h-1}_{m-1}$ and we follow the same discussion about $\dim(B\cdot\lambda)$ and get the same conclusion that the Euler number is 0.

When $h>0$ then (4.5) says that $\nabla^{2m+h}_m = B\cdot\nabla^{2m+h-1}_{m-1} \sqcup T_m\nabla^{m+h}_m$ and we know the alternating sum is 0 of the first term. Concerning the second term, take an arbitrary element $\lambda = [\ell_1,\ell_2,\dots] \in \nabla^{m+h}_m$ with the conditions $\sum_s \ell_s = m$ and $\sum_s s\ell_s = m+h$. Then $T_m\cdot\lambda = [0,\ell 1,\ell_2,\dots]$ and so

$$
\begin{aligned}
&\sum_m (-1)^m \sum_{\lambda\in\nabla^{m+h}_m} \dim(T_m\cdot\lambda) \\
&= \sum_m (-1)^m \sum_{\lambda\in\nabla^{m+h}_m} \binom{\dim \mathfrak{X}^1_1}{\ell 1}\binom{\dim \mathfrak{X}^1_2}{\ell 2}\dots = \sum_{\sum_s \ell_s = \sum_s s\ell_s - h} (-1)^{\sum_s \ell_s}\binom{\dim \mathfrak{X}^1_1}{\ell 1}\binom{\dim \mathfrak{X}^1_2}{\ell 2}\dots \\
&= \sum_{\ell_1} (-1)^{\ell_1}\binom{\dim \mathfrak{X}^1_1}{\ell 1} \sum_{\sum_s \ell_s = \sum_s s\ell_s - h} (-1)^{\sum_{s\neq 1}\ell_s}\binom{\dim \mathfrak{X}^1_2}{\ell 2}\binom{\dim \mathfrak{X}^1_3}{\ell 3}\dots \\
&= 0\ . \qquad \square
\end{aligned}
$$

4.1.2. *Case where the first weight $w=1$*

Assume $w=1$. Then using Corollary 4.1 directly, we have

$$
\mathrm{C}_{m,1,h} = \sum_{\sum_{s=1}^m (h_s+1)=h+2m} \mathfrak{X}^{[m-1,1,0,\dots]}_{(h_1,\dots,h_m)} = \sum_{h_m} \mathrm{SubC}^{(1:(m-1))}_{[h+1-h_m]}\,\Delta\,\mathrm{SubC}^{(2:1)}_{[h_m]}
$$

and $\mathrm{SubC}^{(1:(m-1))}_{[h+1-h_m]}$ is nothing but $\mathrm{C}_{m-1,0,h+1-h_m}$. Thus, we have the following proposition which gives a rule for an expression of $\mathrm{C}_{m,1,h}$ by lower weight chain spaces $\mathrm{C}_{m-1,0,h'}$.

Proposition 4.4. *The chain complex $\{C_{\bullet,1,h}\}$ is non-trivial if $h \geq -(1+\dim\mathfrak{X}^1_0)$, and*

$$
C_{m,1,h} = \sum_{h'} C_{m-1,0,h-h'+1}\Delta\mathfrak{X}^2_{h'} \quad \textit{for} \quad m\geq 1\ . \tag{4.6}
$$

Each degree m of the chain complex is bounded from above by $h+2+2\dim\mathfrak{X}^1_0+\dim\mathfrak{X}^1_1$.

For $\mathbb{R}^n$ with n general, the Euler number of the chain complex $\{C_{\bullet,1,h}\}$ is always 0 for each h.

Proof. From the definition of double weight, $\mathrm{C}_{m,1,h} = \sum \mathfrak{X}^{a_1}_{b_1} \Delta \cdots \Delta \mathfrak{X}^{a_m}_{b_m}$ where $1 \leq a_1 \leq \cdots \leq a_m \leq n$ with $\sum_{i=1}^{m}(a_i - 1) = 1$, and integers $(b_i \geq 0)$ with $\sum_{i=1}^{m}(b_i - 1) = h$. We see directly that $a_i - 1 = 0$ for $i < m$ and $a_m - 1 = 1$. $\sum_{i=1}^{m}(b_i - 1) = h$ implies $\sum_{i=1}^{m-1}(b_i - 1) = h - b_m + 1$. Thus, we get (4.6).

(4.6) implies $\dim \mathrm{C}_{m,1,h} = \sum_{h'} \dim \mathrm{C}_{m-1,0,h-h'+1} \dim \mathfrak{X}^2_{h'}$ for $m \geq 1$.

$$\begin{aligned}
\sum_{m}(-1)^m \dim \mathrm{C}_{m,1,h} &= \sum_{m\geq 1}(-1)^m \sum_{h'} \dim \mathrm{C}_{m-1,0,h-h'+1} \dim \mathfrak{X}^2_{h'} \\
&= -\sum_{h'} \dim \mathfrak{X}^2_{h'} \sum_{m\geq 1}(-1)^{m-1} \dim \mathrm{C}_{m-1,0,h-h'+1} \\
&= \ 0 \quad \text{using Lemma 4.1.}
\end{aligned}$$
□

4.1.3. *Case where the first weight $w = 2$*

Assume $w = 2$. Again, using Corollary 4.1, we have

$$\begin{aligned}
\mathrm{C}_{m,2,h} &= \sum_{\sum_{s=1}^{m}(h_s+1)=h+2m} \mathfrak{X}^{[m-1,0,1,0,\dots]}_{(h_1,\dots,h_m)} + \sum_{\sum_{s=1}^{m}(h_s+1)=h+2m} \mathfrak{X}^{[m-2,2,0,\dots]}_{(h_1,\dots,h_m)} \\
&= \sum \mathrm{SubC}^{(1:(m-1))}_{[h+1-h_m]} \Delta \mathfrak{X}^3_{h_m} + \mathrm{SubC}^{(1:(m-2))}_{[h-h']} \Delta\, \mathrm{SubC}^{(2:2)}_{[h']} \ .
\end{aligned}$$

Thus, we have the following proposition which gives a rule of expression of $\mathrm{C}_{m,2,h}$ by lower weight chain spaces.

Proposition 4.5. *The chain complex $\{C_{\bullet,2,h}\}$ is non-trivial if $h \geq -(2 + \dim \mathfrak{X}^1_0)$, and*

$$C_{m,2,h} = \sum_{h'} C_{m-1,0,h+1-h'} \Delta \mathfrak{X}^3_{h'} \sqcup \sum_{a\leq b} C_{m-2,0,h+2-a-b} \Delta \mathfrak{X}^2_a \Delta \mathfrak{X}^2_b \quad \textit{for} \quad m \geq 2 \ , \tag{4.7}$$

and

$$C_{1,2,h} = \mathfrak{X}^3_{h+1} \ . \tag{4.8}$$

The range of degree m of the chain complex has an upper bound $h + 4 + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$.

We can apply Lemma 4.1 to the chain complex $w = 2$, we have

Proposition 4.6. *For general n, the Euler number of chain complex $\{C_{\bullet,2,h}\}$ is always 0 for each h.*

Proof. We make the alternating sum of the dimensions of chain spaces. First we sum up the terms which involve $\mathfrak{X}^3_\bullet$ as follows:

$$\begin{aligned} A =&(-1)^1 \dim \mathfrak{X}^3_{h+1} + \sum_{m\geq 2} (-1)^m \sum_{h'} \dim \mathrm{C}_{m-1,0,h+1-h'} \dim \mathfrak{X}^3_{h'} \\ =& -\sum_{h'} \dim \mathfrak{X}^3_{h'} \sum_{m\geq 0} (-1)^m \dim \mathrm{C}_{m,0,h+1-h'} = 0 \quad \text{using Lemma 4.1.} \end{aligned}$$

The rest is

$$\begin{aligned} B =& \sum_{m\geq 2} (-1)^m \sum_{a\leq b} \dim \mathrm{C}_{m-2,0,h+2-a-b} \dim(\mathfrak{X}^2_a \Delta \mathfrak{X}^2_b) \\ =& \sum_{a\leq b} \dim(\mathfrak{X}^2_a \Delta \mathfrak{X}^2_b) \sum_{m\geq 2} (-1)^{m-2} \dim \mathrm{C}_{m-2,0,h+2-a-b} = 0 \quad \text{using again Lemma 4.1.} \end{aligned}$$

□

4.1.4. *Case where the first weight is general*

Inspired by Propositions 4.4 and 4.6, we have the following general result which include those propositions as special cases.

Theorem 4.1. *For general n and for general first weight w, the range of degree m of the non-trivial chain complex $\mathrm{C}_{m,w,h}$ is upper bounded as follows:*

$$m \leq h + 2w + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1 \ , \tag{4.9}$$

and the Euler number of chain complex $\{C_{\bullet,w,h}\}$ is 0 for each w and h.

Proof. We have already seen that the above is valid for $w = 0, 1, 2$ in Propositions 4.3, 4.4 and 4.5. So we may assume $w > 2$ and $m > 0$. First, we prove (4.9) by induction. Assume (4.9) holds for each w with $w < \Omega$. From the definition of chain space,

$$\mathrm{C}_{m,\Omega,h} = \sum \mathfrak{X}^{a_1}_{b_1} \Delta \cdots \Delta \mathfrak{X}^{a_m}_{b_m} \tag{4.10}$$

$$\sum_{i=1}^{m} (a_i - 1) = \Omega \quad \text{and} \quad 1 \leq a_1 \leq \cdots \leq a_m \leq n = \dim M \ , \tag{4.11}$$

$$\sum_{i=1}^{m} (b_i - 1) = h \quad \text{where} \quad b_i \geq 0 \ . \tag{4.12}$$

From (4.11) and (4.12), we have $\sum_{i=1}^{m-1} (a_i - 1) = \Omega - (a_m - 1)$ and $\sum_{i=1}^{m-1} (b_i - 1) = h - (b_m - 1)$, and we see that $\mathrm{C}_{m,\Omega,h} = \sum \mathrm{C}_{m-1,\Omega+1-a_m,h+1-b_m} \Delta \mathfrak{X}^{a_m}_{b_m}$. $\mathrm{C}_{m,\Omega,h}$ is non-trvial if and only if some $\mathrm{C}_{m-1,\Omega+1-a_m,h+1-b_m}$ is non-trivial. We may assume that $\Omega > 2$. Then $a_m > 1$ holds, namely $\Omega + 1 - a_m \leq \Omega - 1$ holds and satisfies the assumption of induction. Thus,

$$m - 1 \leq h + 1 - b_m + 2(\Omega + 1 - a_m) + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$$

$$\leq h + 1 + 2(\Omega + 1 - 2) + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$$
$$= h - 1 + 2\Omega + 2\dim \mathfrak{X}^1_0 + \dim \mathfrak{X}^1_1$$

imply (4.9) holds for Ω.

We use the notation (4.1). From Corollary 4.1, the chain space is written by

$$\mathrm{C}_{m,w,h} = \bigoplus_{\substack{\sum_{i=1}^n k_i = m \\ \sum_{i=1}^n i k_i = w+m \\ \sum_{i=1}^n u_i = h}} \mathrm{SubC}^{(1:k_1)}_{[u_1]} \Delta \cdots \Delta \mathrm{SubC}^{(n:k_n)}_{[u_n]} .$$

The first component $\mathrm{SubC}^{(1:k_1)}_{[u_1]}$ is equal to the chain space $\mathrm{C}_{k_1,0,u_1}$ with the first weight 0. Thus

$$\begin{aligned}
&\sum_{m>0} (-1)^m \dim \mathrm{C}_{m,w,h} \\
&= \sum_{\sum_{i=1}^n (i-1)k_i = w} (-1)^{\sum_{s=1}^n k_s} \sum_{u_j} \dim \mathrm{C}_{k_1,0,u_1} \dim \left(\mathrm{SubC}^{(2:k_2)}_{[u_2]} \Delta \cdots \Delta \mathrm{SubC}^{(n:k_n)}_{[u_n]} \right) \\
&= \sum_{\sum_{i=1}^n (i-1)k_i = w} (-1)^{\sum_{s=2}^n k_s} \sum_{u_j} \sum_{k_1} (-1)^{k_1} \dim \mathrm{C}_{k_1,0,u_1} \dim \left(\mathrm{SubC}^{(2:k_2)}_{[u_2]} \Delta \cdots \Delta \mathrm{SubC}^{(n:k_n)}_{[u_n]} \right) .
\end{aligned}$$

Here, we used that the condition $\sum_{i=1}^n (i-1)k_i = w$ does not involve k_1. Now we use Lemma 4.1 and obtain

$$= \sum_{\sum_{i=1}^n (i-1)k_i = w} (-1)^{\sum_{s=2}^n k_s} \sum_{u_j} 0 = 0 .$$

□

Remark 4.2. The pre Lie superalgebra $\mathfrak{g} = \mathfrak{X}^1(M) \oplus \cdots \oplus \mathfrak{X}^n(M)$ is the one we mainly dealt so far. $\overline{\mathfrak{g}} = \mathfrak{X}^0(M) \oplus \mathfrak{X}^1(M) \oplus \cdots \oplus \mathfrak{X}^n(M)$ is a pre Lie superalgebra, which includes $\mathfrak{g}$ as a sub superalgebra, or *extended algebra* of $\mathfrak{g}$. Taking $M = \mathbb{R}^n$ again, we consider the chain spaces coming from $\overline{\mathfrak{g}}$ defined by

$$\overline{\mathrm{C}}_{m+1,w,h} = \sum \mathfrak{X}^0_{h_0}(\mathbb{R}^n) \Delta \mathfrak{X}^{i_1}_{h_1}(\mathbb{R}^n) \Delta \cdots \Delta \mathfrak{X}^{i_m}_{h_m}(\mathbb{R}^n)$$

where $(0-1) + \sum_{s=1}^m (i_s - 1) = w$ and $\sum_{s=0}^m (h_s - 1) = h$. We see easily the following proposition.

Proposition 4.7. *$\partial(\overline{C}_{m+1,w,h}) \subset \overline{C}_{m,w,h}$ and have another double-weighted homology groups. The Euler number of the chain complex $\{\overline{\mathrm{C}}_{\bullet,w,h}\}$ is 0 .*

4.2. *Homology group with representation*

In this subsection, we consider a natural representation of $\mathfrak{g} = \mathfrak{X}^1(M) \oplus \cdots \oplus \mathfrak{X}^n(M)$ for general manifold M. Since the Schouten bracket of $\mathfrak{g}$ with $\mathfrak{X}^0(M) = C^\infty(M)$ lies in $\mathfrak{X}^0(M) \oplus \cdots \oplus \mathfrak{X}^{n-1}(M)$, we regard $\mathfrak{g}$ acting on $\mathfrak{X}^0(M) = C^\infty(M)$ by

$$U \cdot f = [U, f] \mod \mathfrak{g} .$$

Actually, if $U \in \mathfrak{X}^1(M)$ then $U \cdot f = [U, f] = \langle U, df \rangle$ and if $U \in \mathfrak{X}^i(M)$ then $U \cdot f = 0$ for $i > 1$. So we have a representation space $V = \mathfrak{X}^0(M) = C^\infty(M)$ of $\mathfrak{g}$ and the action.

Now we consider $M = \mathbb{R}^n$ and we may study homology groups of the chain spaces $\Delta^m \mathfrak{g} \otimes V$ with the boundary operator ∂_V as introduced in the section 2.

We introduce double-weighted chain spaces using the specialty of the base space $M = \mathbb{R}^n$. The chain spaces are given by

$$\overline{\mathrm{C}}_{m,w,h} = \sum_{\substack{\sum_{s=1}^m (i_s - 1) = w \\ \sum_{s=0}^m (h_s - 1) = h}} \mathfrak{X}^{i_1}_{h_1}(\mathbb{R}^n) \Delta \cdots \Delta \mathfrak{X}^{i_m}_{h_m}(\mathbb{R}^n) \otimes \mathfrak{X}^0_{h_0}(\mathbb{R}^n) \quad (m \geq 0) \tag{4.13}$$

$$= \sum_{h_0} \mathrm{C}_{m,w,h+1-h_0} \otimes \mathfrak{X}^0_{h_0}(\mathbb{R}^n) \quad (m \geq 0)\ , \tag{4.14}$$

where $\mathrm{C}_{\bullet,w,h'}$ are the chain spaces in the trivial module. We easily see the following proposition.

Proposition 4.8. *The double weight is invariant by ∂_V, i.e., $\partial_V \left(\overline{C}_{m,w,h} \right) \subset \overline{C}_{m-1,w,h}$. Thus, we have the double-weighted homology groups $H_{m,w,h}(\mathfrak{g}, V)$ with $\mathfrak{g}$-module V as coefficient.*

Proof. We know that $[\mathfrak{X}^i_h, \mathfrak{X}^{i'}_{h'}] \subset \mathfrak{X}^{i+i'-1}_{h+h'-1}$ and in particular, $[\mathfrak{X}^i_h, \mathfrak{X}^0_{h'}] \subset \mathfrak{X}^{i-1}_{h+h'-1}$.

$$\begin{aligned} &\partial_V(\mathfrak{X}^{i_1}_{h_1} \Delta \cdots \Delta \mathfrak{X}^{i_m}_{h_m} \otimes \mathfrak{X}^0_{h_0}) \\ =&\partial(\mathfrak{X}^{i_1}_{h_1} \Delta \cdots \Delta \mathfrak{X}^{i_m}_{h_m}) \otimes \mathfrak{X}^0_{h_0} \pm \sum_p \mathfrak{X}^{i_1}_{h_1} \Delta \ldots \widehat{\mathfrak{X}^{i_p}_{h_p}} \ldots \mathfrak{X}^{i_m}_{h_m} \otimes [\mathfrak{X}^{i_p}_{h_p}, \mathfrak{X}^0_{h_0}] \end{aligned}$$

Thus, we directly see that the double weight of the first part

$[\mathfrak{X}^{i_p}_{h_p}, \mathfrak{X}^{i_q}_{h_q}] \Delta \ldots \widehat{\mathfrak{X}^{i_p}_{h_p}} \ldots \widehat{\mathfrak{X}^{i_q}_{h_q}} \cdots \otimes \mathfrak{X}^0_{h_0}$ does not change. About the second part, $\mathfrak{X}^{i_1}_{h_1} \Delta \ldots \widehat{\mathfrak{X}^{i_p}_{h_p}} \ldots \mathfrak{X}^{i_m}_{h_m} \otimes [\mathfrak{X}^{i_p}_{h_p}, \mathfrak{X}^0_{h_0}]$ is 0 if $i_p \neq 1$. When $i_p = 1$, the first weight is $\displaystyle\sum_{s \neq p} (i_s - 1) = \sum_{s=1}^m (i_s - 1) = w$ and the second weight is $\displaystyle\sum_{s \neq p} (h_s - 1) + (h_p + h_0 - 1 - 1) = \sum_{s=0}^m (h_s - 1) = h$. □

Due to Lemma 4.1 and Theorem 4.1, we have following result about Euler number of the chain complex $(\overline{\mathrm{C}}_{\bullet,w,h}, \partial_V)$.

Theorem 4.2. *The Euler number of $(\overline{C}_{\bullet,w,h}, \partial_V)$ is 0 for each w and h.*

Proof. From (4.14),

$$\begin{aligned} \sum_{m \geq 0} (-1)^m \dim \overline{\mathrm{C}}_{m,w,h} &= \sum_{m \geq 0} (-1)^m \sum_{h_0} \dim \mathrm{C}_{m,w,h+1-h_0} \dim \mathfrak{X}^0_{h_0} \\ &= \sum_{h_0} \dim \mathfrak{X}^0_{h_0} \sum_{m \geq 0} (-1)^m \dim \mathrm{C}_{m,w,h+1-h_0} \end{aligned}$$

using Theorem 4.1

$$= \sum_{h_0} \dim \mathfrak{X}^0_{h_0} \times 0 = 0 \; . \qquad \square$$

4.3. *Example of pre Lie superalgebra related to a Lie superalgebra*

In Example 1.1, we saw toy models of Lie superalgebra and pre Lie superalgebra. We study the chain complexes of those.

Recall that we have a gradation $\mathfrak{gl}(2) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_0 = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$, $\mathfrak{g}_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $\mathfrak{g}_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Take a basis $\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathfrak{g}_0$, $\mathbf{u}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \mathfrak{g}_1$, $\mathbf{u}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathfrak{g}_2$ with the following bracket relation:

Table 3. the bracket relation and the chain spaces with $\dim$, rank information

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$
$\mathbf{u}_1$	0	$2\mathbf{u}_2$	$-2\mathbf{u}_3$	0
$\mathbf{u}_2$	$-2\mathbf{u}_2$	0	$\mathbf{u}_4$	0
$\mathbf{u}_3$	$2\mathbf{u}_3$	$\mathbf{u}_4$	0	0
$\mathbf{u}_4$	0	0	0	0

m	$w-1$	w	$w+1$
$\mathrm{C}_{m,w}$	$\Delta^{w-2}\mathfrak{g}_1\Delta\mathfrak{g}_2$	$\Delta^{w}\mathfrak{g}_1 \oplus \mathfrak{g}_0\Delta^{w-2}\mathfrak{g}_1\Delta\mathfrak{g}_2$	$\mathfrak{g}_0\Delta^{w}\mathfrak{g}_1$
$\dim$	$w-1$	$2w$	$w+1$
$\dim\partial$	$w-1$	$w+1$	0
Betti	0	0	0

Given a weight w, we consider the chain space for $m = w-1, w, w+1$ then computing the boudary homomorphism, we get dimensions, ranks and Betti numbers as in right above.

The Lie superalgebra in Example 1.1 is sometimes denoted as $\mathfrak{gl}(1|1) = \mathfrak{g}_{[0]} \oplus \mathfrak{g}_{[1]}$. $\mathfrak{g}_{[0]}$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathfrak{g}_{[1]}$ is spanned by $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Those basis satisfy the following bracket relations: Depending on the weight

Table 4. the bracket relation

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	$\mathbf{v}_3$	$-\mathbf{v}_4$
$\mathbf{v}_2$	0	0	$-\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_3$	$-\mathbf{v}_3$	$\mathbf{v}_3$	0	$\mathbf{v}_1 + \mathbf{v}_2$
$\mathbf{v}_4$	$\mathbf{v}_4$	$-\mathbf{v}_4$	$\mathbf{v}_1 + \mathbf{v}_2$	0

to be even or odd, we have

$$\begin{aligned} \mathrm{C}_{2m,[0]} &= \Delta^{2m}\mathfrak{g}_{[1]} \oplus \Delta^2\mathfrak{g}_{[0]}\Delta^{2m-2}\mathfrak{g}_{[1]} \; , & \mathrm{C}_{2m+1,[0]} &= \mathfrak{g}_{[0]}\Delta^{2m}\mathfrak{g}_{[1]} \; , \\ \mathrm{C}_{2m,[1]} &= \mathfrak{g}_{[0]}\Delta^{2m-1}\mathfrak{g}_{[1]} \; , & \mathrm{C}_{2m+1,[1]} &= \Delta^{2m+1}\mathfrak{g}_{[1]} \oplus \Delta^2\mathfrak{g}_{[0]}\Delta^{2m-1}\mathfrak{g}_{[1]} \; . \end{aligned}$$

Denote $\Delta^a\mathbf{v}_3\Delta^b\mathbf{v}_4$ by $F(a,b)$. Then

$$\partial(F(a,b)) = ab(\mathbf{v}_1 + \mathbf{v}_2)\Delta F(a-1, b-1)$$

$$\begin{aligned}\partial(\mathbf{v}_1\Delta F(a,b)) &= -ab\mathbf{v}_1\Delta\mathbf{v}_2\Delta F(a-1,b-1)+(a-b)F(a,b)\\ \partial(\mathbf{v}_2\Delta F(a,b)) &= ab\mathbf{v}_1\Delta\mathbf{v}_2\Delta F(a-1,b-1)-(a-b)F(a,b)\\ \partial(\mathbf{v}_1\Delta\mathbf{v}_2\Delta F(a,b)) &= (a-b)\mathbf{v}_1\Delta\mathbf{v}_2\Delta F(a,b)\end{aligned}$$

Table 5. even cases

$\bullet$	$\dots$	$2m-1$	$2m$	$2m+1$
$\dim \mathrm{C}_{\bullet,[0]}$	$\dots$	$2(2m-1)$	$2(2m)$	$2(2m+1)$
$\dim \partial$	$2m-1$	$2m-1$	$2m+1$	$2m+1$
Betti	$\dots$	0	0	0

Table 6. odd cases

$\bullet$	$2m-2$	$2m-1$	$2m$	$2m+1$
$\dim \mathrm{C}_{\bullet,[1]}$	$2(2m-2)$	$2(2m-1)$	$2(2m)$	$2(2m+1)$
$\dim \partial$	$2(m-1)$	$2m$	$2m$	$2(m+1)$
Betti	$\dots$	0	0	0

5. Betti Numbers of Homology Groups of Concrete pre Lie Superalgebras

So far, we studied the chain spaces $\mathrm{C}_{m,w,h}$ for fixed space dimension n and double weight (w,h). As stated in Remark 3.1, we may find all Poisson structures in the second homology group of pre Lie superalgebra of tangent bundle of M with the Schouten bracket. Thus, it is interesting to study the second and/or the third homology group. But, it seems hard to attack to general manifold M. So, again we consider the pre Lie superalgebra of multi vector fields on $\mathbb{R}^n$ with homogeneous polynomial coefficients. We study not only the second Betti number but also Betti numbers of general degree.

In pre Lie superalgebra theory, recursive formulae of the boundary operator are given in two ways as below: one is given by using right action and the other is given by left action.

$$\begin{aligned}&\partial(A_1\Delta\cdots\Delta A_{m+1})\\ &=\partial(A_1\Delta\cdots\Delta A_m)\Delta A_{m+1}+(-1)^{m+1}(A_1\Delta\cdots\Delta A_m)^{A_{m+1}}\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}&(A_1\Delta\cdots\Delta A_m)^{A_{m+1}}\\ &=\sum_{i=1}^{m}(-1)^{a_{m+1}\sum_{s=i+1}^{m}a_s}A_1\Delta\cdots\Delta[A_i,A_{m+1}]\Delta\cdots\Delta A_m\ .\end{aligned}\tag{5.2}$$

$$\begin{aligned}&\partial(A_0\Delta A_1\Delta\cdots\Delta A_m)\\ &=-A_0\Delta\partial(A_1\Delta\cdots\Delta A_m)+A_0\cdot(A_1\Delta\cdots\Delta A_m)\end{aligned}\tag{5.3}$$

where

$$A_0\cdot(A_1\Delta\cdots\Delta A_m)$$

$$= \sum_{i=1}^{m} (-1)^{a_0 \sum_{s<i} a_s} A_1 \Delta \cdots \Delta [A_0, A_i] \Delta \cdots \Delta A_m) \ , \tag{5.4}$$

for each homogeneous elements $A_i \in \mathfrak{g}_{a_i}$.

In lower degree, the boundary operator is given as below:

$$\partial(A \Delta B) = [A, B] \tag{5.5}$$

$$\partial(A \Delta B \Delta C) = -A \Delta [B, C] + [A, B] \Delta C + (-1)^{ab} B \Delta [A, C] \tag{5.6}$$

for each homogeneous elements $A \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $C \in \mathfrak{g}_c$.

If we will handle Poisson structures on $\mathbb{R}^n$ by homology theory of pre Lie superalgebra, then Remark 3.1 says we will deal with $\{\mathrm{C}_{\bullet, w=2, h}\}$, where

$$\begin{aligned}
\mathrm{C}_{1,2,h} &= \mathfrak{X}^3_{h+1} \ , \\
\mathrm{C}_{2,2,h} &= \sum_{a+b=h+2} \mathfrak{X}^1_a \Delta \mathfrak{X}^3_b + \sum_{a+b=h+2} \mathfrak{X}^2_a \Delta \mathfrak{X}^2_b \ , \\
\mathrm{C}_{3,2,h} &= \sum_{c+a+b=h+2+1} \mathfrak{X}^1_c \Delta \mathfrak{X}^1_a \Delta \mathfrak{X}^3_b + \sum_{c+a+b=h+2+1} \mathfrak{X}^1_c \Delta \mathfrak{X}^2_a \Delta \mathfrak{X}^2_b \ , \\
&\vdots
\end{aligned}$$

Remark 5.1. Since $\mathrm{C}_{m,w,h} = \sum_{\substack{\sum_{i=1}^m a_i = w+m, \\ \sum_{i=1}^m b_i = h+m}} \mathfrak{X}^{a_1}_{b_1} \Delta \mathfrak{X}^{a_2}_{b_2} \Delta \cdots \Delta \mathfrak{X}^{a_m}_{b_m}$ in general, if $\mathrm{C}_{m,w,h} \neq (0)$ then $a_i \leq n$ for each i, and so $\sum_{i=1}^m a_i \leq mn$. Thus, $w \leq m(n-1)$. Namely, w is bounded from above by the dimension n and the degree m of the chain space.

On $M = \mathbb{R}^n$, we have a special vector field $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \in \mathfrak{X}^1_1$ (called Euler vector field). It is known that if f is h-homogeneous polynomial, then $[E, f] = hf$. If $D \in \mathfrak{X}^p_0$, then $[E, D] = -pD$ holds. In general, we have the following lemma

Lemma 5.1. *For each $U \in \mathfrak{X}^p_h$,*

$$E \cdot U = [E, U] = (-p + h) U \tag{5.7}$$

holds. In fact, the action of E is divided into two parts:

$$\sum_{k=1}^{n} x_k [(\frac{\partial}{\partial x_k}), U] = hU \ , \tag{5.8}$$

and

$$\sum_{k=1}^{n} (\frac{\partial}{\partial x_k}) \wedge [x_k, U] = -pU \ . \tag{5.9}$$

Thus,

$$E \cdot W = (-w + h) W \quad (\forall W \in C_{m,w,h}) \ . \tag{5.10}$$

Proof. In order to prove (5.10), using (5.4), we have

$$E \cdot \sum_i A^i_1 \Delta \cdots \Delta A^i_m = \sum_i \sum_{k=1}^{m} A^i_1 \Delta \cdots \Delta [E, A^i_k] \Delta \cdots \Delta A^i_m$$

$$
\begin{aligned}
&= \sum_i \sum_{k=1}^m A_1^i \Delta \cdots \Delta(-w(A_k^i) + \bar{h}(A_k^i)) A_k^i \Delta \cdots \Delta A_m^i \\
&= \sum_i \sum_{k=1}^m (-w(A_k^i) + \bar{h}(A_k^i)) A_1^i \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_m^i \\
&= \sum_i (-w + h) A_1^i \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_m^i \\
&= (-w + h) \sum_i A_1^i \Delta \cdots \Delta A_k^i \Delta \cdots \Delta A_m^i \ .
\end{aligned}
$$

□

Using Lemma above, we have the following proposition.

Proposition 5.1. *Define a map* $\phi : C_{m,w,h} \to C_{m+1,w,h}$ *by* $\phi(U) = E\Delta U$. *Then we have*

$$\partial \circ \phi + \phi \circ \partial = (-w + h)\, id \ . \tag{5.11}$$

Proof. Take $W \in \mathrm{C}_{m,w,h}$, then we have

$$\partial(\phi W) = \partial(E\Delta W) \overset{(5.3)}{=} -E\Delta\partial W + E \cdot W \overset{(5.10)}{=} -\phi(\partial W) + (-w + h)W \ .$$

□

Directly from this proposition we have the following theorem.

Theorem 5.1 (m-th Betti number). *Each* m*-th Betti numbers of* (w, h)*-weighted chain complex* $\{C_{\bullet,w,h}\}$ *is 0 if* $w \neq h$.

Proof. Take a cycle $W \in \mathrm{C}_{m,w,h}$. The proposition above yields

$$(-w + h)W = \partial(\phi(W)) + \phi(\partial U) = \partial(\phi(W))$$

and

$$W = \frac{1}{-w + h}\partial(E\Delta W) \quad \text{if} \quad w \neq h \ .$$

This means the result. □

Remark 5.2. When $w = h$, Theorem 5.1 says $E\Delta U$ is a cycle if U is a cycle in $\mathrm{C}_{m,w,h}$.

Remark 5.3. We have the table of Betti numbers of $\{\mathrm{C}_{\bullet,0,0}\}$ of $\mathbb{R}^2$ in Example 4.1, which shows non-trivial Betti numbers: $b_0 = 1, b_5 = 2, b_7 = 1, b_8 = 2$.

For the first Betti number, we have the following result without any restriction on w, h.

Theorem 5.2 (1st Betti number). *The first Betti number of* (w, h)*-weighted chain complex* $\{C_{\bullet,w,h}\}$ *is 0 for each double weight* (w, h).

Proof. Fix a general weight (w, h). $\mathrm{C}_{1,w,h} = \mathfrak{X}_{h+1}^{w+1}$ and $\mathrm{C}_{2,w,h} = \sum\limits_{\substack{p+q=2+w \\ a+b=2+h}} \mathfrak{X}_a^p \Delta \mathfrak{X}_b^q$.

Take $\forall U \in \mathfrak{X}^{w+1}_{h+1}$. Then $(\dfrac{\partial}{\partial x_k})\Delta(x_k U) \in \mathfrak{X}^1_0 \Delta \mathfrak{X}^{w+1}_{h+2} \subset \mathrm{C}_{2,w,h}$. Now we see

$$\begin{aligned}\partial((\frac{\partial}{\partial x_k})\Delta x_k U) =&[(\frac{\partial}{\partial x_k}), x_k U] = U + x_k[(\frac{\partial}{\partial x_k}), U]\\ \sum_{k=1}^{n} \partial((\frac{\partial}{\partial x_k})\Delta x_k U) =&nU + \sum_{k=1}^{n} x_k[(\frac{\partial}{\partial x_k}), U] = (n+1+h)U\ . \end{aligned}$$
□

We have the same result about the second Betti number when $w = h = 0$.

Proposition 5.2. *The 2nd Betti number is zero when $w = h = 0$.*

Proof. Since $\mathrm{C}_{2,0,0} = \sum_{a+b=0+2, a\leq b} \mathfrak{X}^1_a \Delta \mathfrak{X}^1_b = \mathfrak{X}^1_0 \Delta \mathfrak{X}^1_2 + \mathfrak{X}^1_1 \Delta \mathfrak{X}^1_1$, a general cycle $T \in \mathrm{C}_{2,0,0}$ is given by $T = \sum_i A_i \Delta B_i + \sum_{j<\ell} p^{j,\ell} Y_j \Delta Y_\ell$ where

$$A_i \in \mathfrak{X}^1_0\ ,\ B_i \in \mathfrak{X}^1_2\ ,\ Y_j \in \mathfrak{X}^1_1\ ,\ p^{j,\ell} + p^{\ell,j} = 0$$

and with the cycle condition $\partial(\sum_i A_i \Delta B_i + \sum_{j<\ell} p^{j,\ell} Y_j \Delta Y_\ell) = 0$, i.e., $\sum_i [A_i, B_i] + \sum_{j<\ell} p^{j,\ell}[Y_j, Y_\ell] = 0$.

Consider $\displaystyle\sum_i (\frac{\partial}{\partial x_k})\Delta A_i \Delta(x_k B_i) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta Y_j \Delta(x_k Y_\ell) \in \mathrm{C}_{3,0,0}$. We compute its boundary image.

$$\begin{aligned}&\partial\left(\sum_i (\frac{\partial}{\partial x_k})\Delta A_i \Delta(x_k B_i) + \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta Y_j \Delta(x_k Y_\ell)\right)\\
=&-\sum_i (\frac{\partial}{\partial x_k})\Delta[A_i, x_k B_i] + \sum_i [(\frac{\partial}{\partial x_k}), A_i]\Delta(x_k B_i) + \sum_i A_i \Delta[(\frac{\partial}{\partial x_k}), x_k B_i]\\
&-\sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta[Y_j, x_k Y_\ell] + \sum_{j<\ell} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta[(\frac{\partial}{\partial x_k}), x_k Y_\ell]\\
=&-\sum_i (\frac{\partial}{\partial x_k})\Delta([A_i, x_k]B_i + x_k[A_i, B_i]) + \sum_i [(\frac{\partial}{\partial x_k}), A_i]\Delta(x_k B_i)\\
&\quad+\sum_i A_i \Delta[(\frac{\partial}{\partial x_k}), x_k A_i] - \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell + x_k[Y_j, Y_\ell])\\
&\quad+\sum_{j<\ell} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta[(\frac{\partial}{\partial x_k}), x_k Y_\ell]\end{aligned}$$

from cycle condition, we have

$$\begin{aligned}=&-\sum_i (\frac{\partial}{\partial x_k})\Delta([A_i, x_k]B_i) + \sum_i [(\frac{\partial}{\partial x_k}), A_i]\Delta(x_k B_i) + \sum_i A_i \Delta[(\frac{\partial}{\partial x_k}), x_k B_i]\\
&-\sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell) + \sum_{j<\ell} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta[(\frac{\partial}{\partial x_k}), x_k Y_e ll]\end{aligned}$$

since $[A_i, x_k]$ are constant number, we have

$$
\begin{aligned}
=&-\sum_i [A_i, x_k](\frac{\partial}{\partial x_k})\Delta B_i + 0 + \sum_i A_i \Delta[(\frac{\partial}{\partial x_k}), x_k B_i] - \sum_{j<\ell} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell) \\
&+ \sum_{j<\ell} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta[(\frac{\partial}{\partial x_k}), x_k Y_\ell]\ .
\end{aligned}
$$

Now, summing up by k, we have

$$
\begin{aligned}
&\sum_k \partial\left(\sum_i (\frac{\partial}{\partial x_k})\Delta A_i \Delta(x_k B_i) + \sum_j (\frac{\partial}{\partial x_k})\Delta Y_j \Delta(x_k Y_\ell)\right) \\
=&-\sum_i A_i\Delta B_i + \sum_{i,k} A_i \Delta(B_i + x_k[(\frac{\partial}{\partial x_k}), B_i]) \\
&+\sum_{j,k}\left(-(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell) + [(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + Y_j\Delta(Y_\ell + x_k[(\frac{\partial}{\partial x_k}), Y_\ell])\right) \\
=&-\sum_i A_i \Delta B_i + \sum_i A_i \Delta((n+2)B_i) - \sum_{j<\ell,k} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell) \\
&+\sum_{j<\ell,k} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell) + \sum_{j<\ell} p^{j,\ell} Y_j \Delta((n+1)Y_\ell) \\
=&(n+1)(\sum_i A_i \Delta B_i + \sum_{j<\ell} p^{j,\ell} Y_j \Delta Y_\ell) \\
&\quad - \sum_{j<\ell,k} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta([Y_j, x_k]Y_\ell) + \sum_{j<\ell,k} p^{j,\ell}[(\frac{\partial}{\partial x_k}), Y_j]\Delta(x_k Y_\ell)\ .
\end{aligned}
$$

It is enough to show the sum of the last two terms vanishes and we can do it as follows: Since $\mathfrak{X}^1_1 \ni Y_j = \sum_{k,\ell} Y_j^{k,\ell} x_\ell (\frac{\partial}{\partial x_k})$ where $Y_j^{k,\ell}$ are constant, $[Y_j, x_k]Y_\ell = Y_j^k Y_\ell = \sum_t Y_j^{k,t} x_t Y_\ell$ and $[(\frac{\partial}{\partial x_k}), Y_j](\frac{\partial}{\partial x_s}) = \sum_s Y_j^{s,k}(\frac{\partial}{\partial x_s})$. Thus, 2nd term+3rd term becomes

$$
\begin{aligned}
&-\sum_{j<\ell,k} p^{j,\ell}(\frac{\partial}{\partial x_k})\Delta(\sum_t Y_j^{k,t} x_t Y_\ell) + \sum_{j<\ell,k} p^{j,\ell}\sum_s Y_j^{s,k}(\frac{\partial}{\partial x_s})\Delta(x_k Y_\ell) \\
=&-\sum_{j<\ell} p^{j,\ell}\sum_k (\frac{\partial}{\partial x_k})\Delta(\sum_t Y_j^{k,\ell} x_\ell Y_\ell) + \sum_{j<\ell} p^{j,\ell}\sum_s Y_j^{s,k}(\frac{\partial}{\partial x_s})\Delta(x_k Y_\ell) \\
=&-\sum_{j<\ell} p^{j,\ell}\sum_k (\frac{\partial}{\partial x_k})\Delta(\sum_t Y_j^{k,\ell} x_\ell Y_\ell) + \sum_{j<\ell} p^{j,\ell}\sum_s (\frac{\partial}{\partial x_s})\Delta(Y_j^{s,k} x_k Y_\ell) \\
=&0\ .
\end{aligned}
$$

□

Remark 5.4. We expect to know the second Betti number of the chain complex $\{\mathbf{C}_{\bullet,w,w}\}$ for $w > 0$. We know the second Betti number is 0 for lower n.

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