Higher weight Gel'fand-Kalinin-Fuks classes of formal Hamiltonian vector fields of symplectic $\mathbb{R}^2$

Kentaro Mikami* Yasuhara Nakae† Hiroki Kodama‡

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1 Introduction

There are many works about Gel'fand-Kalinin-Fuks cohomology. In Japan, the authors had known two works Shinya Metoki's Doctoral Thesis [4], which we can see in Prof. T. Tsuboi's Web Page, and Masashi Takamura's work [6]. At Tambara meeting on October 2009, Prof. S. Morita gave a very interesting talk, that is based on [3] D. Kotschick and S. Morita; "The Gel'fand-Kalinin-Fuks class and characteristic classes of transversely symplectic foliations, Xarchive October 2009".
The relative Gel'fand-Fuks cohomologies in [3] have two parameters, one is degree and the other is weight. If our observation is correct, they handled those cohomologies with weight 10 or so. we were ambitious to compute more higher weight cases. In this paper, we recall fundamental facts on Hamiltonian formalism, in particular, careful review of group actions. And then we concentrate about the weight. We see that the weight corresponds to Young diagrams, and we show the generating function for weight. We make use of Representation theory of $Sp(2n)$ and the irreducible representations are parameterized by the Young diagrams of height at most n. It is not clear that the appearance of Young diagrams for weight is incident or accident. We demonstrate our computer manipulations in easy cases, and then show our main result in the final section without an ordinary proof.

2 Preliminaries

2.1 Lie algebra Cohomology and Gel'fand-Kalinin-Fuks cohomology

Take a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Let (ρ, W) be a representation of $\mathfrak{g}$. Namely, ρ is a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra $\mathrm{End}(W)$. For each $k \in \mathbb{Z}_{\geq 0}$,

$$C^k(\mathfrak{g}) := \{\sigma : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k-\text{times}} \to W \mid \text{alternative and } \mathbb{R}\text{-multilinear}\}.$$

For each k-th cochain $\sigma \in C^k(\mathfrak{g})$, we define

$$\begin{aligned}(d\,\sigma)(X_0, \ldots, X_k) := &\sum_{i=0}^{k} (-1)^i \rho(X_i)\sigma(\ldots \widehat{X_i} \ldots) \\ &+ \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j] \ldots \widehat{X_i} \ldots \widehat{X_j} \ldots)\end{aligned}$$

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†Department of Computer Science and Engineering Akita University
‡Graduate School of Mathematical Sciences, the University of Tokyo, partially supported by "FIRST Program" of JSPS.

it is known that d satisfies $d^2 = 0$ and defines the cohomology groups of a Lie algebra $\mathfrak{g}$ with respect to (ρ, W). In this paper, hereafter we only deal with the trivial representation of Lie algebra, i.e., $W = \mathbb{R}$ and $\rho = 0$.
Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Define

$$C^m(\mathfrak{g}, \mathfrak{k}) := \{\sigma \in C^m(\mathfrak{g}) \mid i_X\sigma = 0, i_X d\,\sigma = 0 \quad (\forall X \in \mathfrak{k})\}$$

and we get the relative cohomology groups $H^m(\mathfrak{g}, \mathfrak{k})$. Let K be a Lie group of $\mathfrak{k}$. Then we also consider

$$C^m(\mathfrak{g}, K) := \{\sigma \in C^m(\mathfrak{g}) \mid i_X\sigma = 0 \ (\forall X \in \mathfrak{k}), Ad_k^*\sigma = \sigma \ (\forall k \in K)\}$$

and we get the relative cohomology groups $H^m(\mathfrak{g}, K)$. If K is connected, those are identical. If K is a closed subgroup of G, then $C^\bullet(\mathfrak{g}, K) = \Lambda^\bullet(G/K)^G$ (the exterior algebra of G-invariant differential forms on G/K).
Take a differentiable manifold M and consider the space $\mathfrak{X}(M)$ of vector fields of M. Then $\mathfrak{X}(M)$ forms a Lie algebra by Jacobi-Lie bracket. Thus, we can consider the Lie algebra cohomology of $\mathfrak{X}(M)$. But, the cochain complex is huge, so we add some restriction, "continuity" by C^∞-topology. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{X}(M)$. For each $k \in \mathbb{Z}_{\geq 0}$,

$$C^k(\mathfrak{g}) := \{\sigma : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k-\text{times}} \to \mathbb{R} \mid \text{continuous, alternative and } \mathbb{R}\text{-multilinear}\}.$$

For each k-th cochain $\sigma \in C^k(\mathfrak{g})$, we define

$$(d\,\sigma)(X_0, \ldots, X_k) := \sum_{i<j} (-1)^{i+j} \sigma([X_i, X_j] \ldots \widehat{X_i} \ldots \widehat{X_j} \ldots)$$

it is known that d satisfies $d^2 = 0$ and defines the cohomology group, called Gel'fand-Kalinin-Fuks cohomology of $\mathfrak{X}(M)$. There are also relative versions.

2.2 Recall of Hamilton formalism

Let (M, ω) be a symplectic manifold, namely, ω is a nondegenerate closed 2-form on M, and so dim$M$ is even. We denote the group of symplectic automorphisms of (M, ω) by $Aut(M, \omega)$. By $\mathfrak{aut}(M, \omega)$, we denote the space of vector fields on M satisfying $\mathcal{L}_X\omega = 0$ (infinitesimal automorphism of ω). Since $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ holds, $\mathfrak{aut}(M, \omega)$ forms a Lie algebra.
For $\forall f$ on M, we have the Hamiltonian vector field on (M, ω) defined by $\omega(\mathcal{H}_f, \cdot) = df$, and the Poisson bracket given by $\{f, g\} := \omega(\mathcal{H}_f, \mathcal{H}_g)$. Since $\mathcal{H}_{\{f,g\}} = -[\mathcal{H}_f, \mathcal{H}_g]$ holds, the Hamiltonian vector fields of (M, ω) form a Lie subalgebra of $\mathfrak{aut}(M, \omega)$. The correspondence $f \mapsto -\mathcal{H}_f$ is a Lie algebra homomorphism and the kernel is $\mathbb{R}$ when M is connected.
It holds $\varphi\mathcal{H}_f = \mathcal{H}_{f\circ\varphi^{-1}}$ for each $\varphi \in Aut(M, \omega)$, and $\{f, g\} \circ \varphi = \{f \circ \varphi, g \circ \varphi\}$ for $f, g \in C^\infty(M)$.
Let K be a Lie subalgebra of $Aut(M, \omega)$ with its Lie algebra $\mathfrak{k}$. For each $\xi \in \mathfrak{k}$, the fundamental vector field on M, say ξ_M, is defined by $\xi_M := \dfrac{d}{dt}\exp(t\xi)_{M|t=0}$. They satisfy $\mathcal{L}_{\xi_M}\omega = 0$ and $[\xi, \eta]_M = -[\xi_M, \eta_M]$, thus form a subalgebra of $\mathfrak{aut}(M, \omega)$. The momentum mapping J (if exists) is a map from $M \to \mathfrak{k}^*$ satisfying

$$d\hat{J}(\xi) = \omega(\xi_M, \cdot), \quad \text{i.e.,} \quad \xi_M = \mathcal{H}_{\hat{J}(\xi)}$$

where $\hat{J}(\xi)$ is defined as $\langle \hat{J}(\xi), m\rangle := \langle \xi, J(m)\rangle$ for $\xi \in \mathfrak{k}, m \in M$. The above definition means J provides with a Hamiltonian potential for each fundamental vector field of K and $\hat{J}(\xi)$ is defined as $\langle \hat{J}(\xi), m\rangle := \langle \xi, J(m)\rangle$ for $\xi \in \mathfrak{k}, m \in M$. If J is K-equivariant, i.e., $J(a \cdot m) = Ad^*_{a^{-1}}(J(m))$ for $\forall a \in K, m \in M$, then $\hat{J}([\xi, \eta]) = \{\hat{J}(\xi), \hat{J}(\eta)\}$ hold for $\xi, \eta \in \mathfrak{k}$, and vice versa if K is connected.

In local, by Darboux's theorem we always have a local coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ such that $\omega(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i}) = -\omega(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^i}) = 1$, the others are 0 and so the Hamiltonian vector field is given by

$$\mathcal{H}_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

and the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial(f, g)}{\partial(q^i, p_i)} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

If $M = (\mathbb{R}^{2n}, \text{linear symplectic structure})$, the space of the Hamiltonian vector fields of (M, ω) coincides with $\mathfrak{aut}(M, \omega)$, because of the space is connected and 1-connected. Let K be the linear symplectic group of $(\mathbb{R}^{2n}, \omega)$ with its Lie algebra $\mathfrak{k}$, i.e., $K = Sp(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{sp}(2n, \mathbb{R})$. The equivariant (co-)momentum mapping is given by

$$\hat{J}(\xi)(q, p) = \frac{-1}{2} (q, p) \, (\text{matrix representation of } \omega) \, \xi \begin{pmatrix} q \\ p \end{pmatrix}$$

here q is the natural coordinate of $\mathbb{R}^n$, $\hat{J}$ is a Lie algebra homomorphism from the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ into $C^\infty(M)$, with the Poisson bracket. The Hamilton potential of Hamiltonian vector field $[\xi_M, \mathcal{H}_f]$ is given by $-\{\hat{J}(\xi), f\}$, because of $[\xi_M, H_f] = [\mathcal{H}_{\hat{J}(\xi)}, \mathcal{H}_f] = -\mathcal{H}_{\{\hat{J}(\xi), f\}}$.

3 Cochain complexes, weight and relative cochains

3.1 Cochain complexes

When $M = \mathbb{R}^{2n}$, the Hamiltonian potential for each Hamiltonian vector field is unique up to constant, we consider the Lie subalgebra $\mathfrak{ham}_{2n}$ of Hamiltonian vector fields is isomorphic with the formal polynomial space. Then the space is Lie algebra isomorphic with the space of $\mathbb{R}[[q^1, \dots, q^n, p_1, \dots, p_n]]/\mathbb{R}$ quotiented by $\mathbb{R}$, where the Lie bracket is given by the Poisson bracket.
We see that

$$\mathbb{R}[[q^1, \dots, q^n, p_1, \dots, p_n]]/\mathbb{R} = \bigoplus_{\ell=1}^{\infty} S^\ell$$

where S^ℓ is the ℓ-th symmetric power of $q^1, \dots, q^n, p_1, \dots, p_n$. It holds $\{S^k, S^\ell\} \subset S^{k+\ell-2}$, because the Poisson bracket satisfy $\{q^i, p_j\} = -\{p_j, q^i\} = \delta^i_j$ othoers are 0, and $\hat{J} : \mathfrak{sp}(2n, \mathbb{R}) \longrightarrow S^2$ is a Lie algebra isomorphism.
If we denote the dual of S^ℓ by $\mathfrak{S}_\ell$, then we see the first cochain complex is

$$C^1(\mathfrak{ham}_{2n}) \cong {\mathfrak{ham}_{2n}}^* = \bigoplus_{\ell=1}^{\infty} \mathfrak{S}_\ell$$

and the second cochain complex is

$$C^2(\mathfrak{ham}_{2n}) \cong {\mathfrak{ham}_{2n}}^* \wedge {\mathfrak{ham}_{2n}}^* = \bigoplus_{\ell=1}^{\infty} \mathfrak{S}_\ell \wedge \bigoplus_{k=1}^{\infty} \mathfrak{S}_k = \bigoplus_{1 \le k \le \ell} \mathfrak{S}_k \wedge \mathfrak{S}_\ell$$

and so on.

3.2 Weight of cochains

Definition 1 (cf.[3]) Define the **weight** of each element of $\mathfrak{S}_\ell$ by $\ell - 2$.

For each element of $\mathfrak{S}_{\ell_1} \wedge \mathfrak{S}_{\ell_2} \wedge \cdots \mathfrak{S}_{\ell_s}$, define its **weight** by $\sum_{i=1}^{s} (\ell_i - 2)$.

Remark 3.1 Let $\sigma \in \mathfrak{S}_\ell$ be a 1-cochain. Since $(d\,\sigma)(f_0, f_1) = -\langle \sigma, \{f_0, f_1\}\rangle$, the contribution of σ is when the case of $\{f_0, f_1\} \in S^\ell$. If $f_0 \in S^{p_0}$ and $f_1 \in S^{p_1}$, then it must hold $p_0 + p_1 - 2 = \ell$, namely, $d\,\mathfrak{S}_\ell \subset \sum_{p_0+p_1=2+\ell} \mathfrak{S}_{d_0} \wedge \mathfrak{S}_{d_1}$. Similarly, we see that

$$d\,(\mathfrak{S}_k \wedge \mathfrak{S}_\ell) \subset \sum_{p_0+p_1+p_2=k+\ell+2} \mathfrak{S}_{p_0} \wedge \mathfrak{S}_{p_1} \wedge \mathfrak{S}_{p_2}.$$

$p_0 + p_1 - 2 = \ell$ is equivalent to $(p_0 - 2) + (p_1 - 2) = \ell - 2$, and $p_0 + p_1 + p_2 = k + \ell + 2$ is $(p_0 - 2) + (p_1 - 2) + (p_2 - 2) = (k - 2) + (\ell - 2)$. These show the reason of the definition of **weight** above. And we also see that the coboundary operator d preserve the weight, namely if a cochain σ is of weight w, then $d\,(\sigma)$ is also of weight w.

Now we can decompose the cochain complex by the weight w as follows:

$$C^\bullet_{GF}(\mathfrak{ham}_{2n})_w := \sum_{w\text{-condition}} \Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \cdots$$

where w-condition is $\sum_{j=1}^{\infty}(j-2)k_j = w$. $C^\bullet_{GF}(\mathfrak{ham}_{2n}) \cong \sum_{w=-2n}^{\infty} C^\bullet_{GF}(\mathfrak{ham}_{2n})_w$. Since the coboundary operator d preserves the weights and so we have the natural splitting of cohomology groups like

$$H^\bullet_{GF}(\mathfrak{ham}_{2n}) \cong \sum_{w=-2n}^{\infty} H^\bullet_{GF}(\mathfrak{ham}_{2n})_w$$

In order to investigate $H^m_{GF}(\mathfrak{ham}_{2n})_w$, we have to handle

$$C^m_{GF}(\mathfrak{ham}_{2n})_w := \sum \Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \cdots$$

$\sum_{j=1}^{\infty} k_j = m$ and $\sum_{j=1}^{\infty}(j-2)k_j = w$. We have to be careful of no contribution of (k_2)-term to w. $\mathfrak{ham}^0_{2n}$ be the space of the Hamiltonian vector fields without constant vector field. Then we have

$$\mathfrak{ham}_{2n} \cong \mathop{\oplus}_{\ell=1}^{\infty} S^\ell, \quad \mathfrak{ham}^0_{2n} \cong \mathop{\oplus}_{\ell=2}^{\infty} S^\ell\ .$$

We look for the relative $H^\bullet_{GF}(\mathfrak{ham}^0_{2n}, Sp(\mathbb{R}^{2n}, \omega))_w$.

Remark 3.2 ([3]) If w is odd then $C^\bullet_{GF}(\mathfrak{ham}_{2n}, Sp(2n, \mathbb{R}))_w = \{0\}$.

$\begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} =: -\mathrm{Id}$ is an element of $Sp(2n, \mathbb{R})$ and each cochain σ should be invariant under the action of $-\mathrm{Id}$.

$$\forall \sigma \in C^m_{GF}(\mathfrak{ham}_{2n})_w := \sum \Lambda^{k_1}\mathfrak{S}_1 \otimes \Lambda^{k_2}\mathfrak{S}_2 \otimes \cdots \quad \text{, where} \quad \sum_{j=1}^{\infty} k_j = m \quad \text{and} \quad \sum_{j=1}^{\infty}(j-2)k_j = w$$

we see that

$$\begin{aligned}\sigma =&(-\mathrm{Id}) \cdot \sigma = (-1)^{k_1}(+1)^{k_2}(-1)^{k_3}(+1)^{k_4} \cdots \sigma \\ =&(-1)^{k_1+k_3+k_5+k_7+\cdots}\sigma = (-1)^{-k_1+k_3+3k_5+5k_7+\cdots}\sigma \\ =&(-1)^{-k_1+0k_2+k_3++2k_4 3k_5+4k_6+5k_7+\cdots}\sigma = (-1)^w\sigma\end{aligned}$$

thus $\sigma = 0$ if w is odd. ∎

Remark 3.3 When $n = 1$, the *type* of Metoki ([4]) and the *weight* here are related by 2 times of *type* is equal to *weight*.

3.3 Relativity

On our original Lie algebra of Hamiltonian vector fields $\mathcal{H}_f$ of polynomial potential functions f, we have the natural group action of $Sp(2n,\mathbb{R})$:

$$a \cdot \mathcal{H}_f = \mathcal{H}_{f\circ a^{-1}}, \qquad a \cdot f := f \circ a^{-1} \ .$$

Definition 2 On the exterior algebra of the space of polynomial functions of $\mathbb{R}^{2n}$, the group $Sp(\mathbb{R}^{2n},\omega)$ acts naturally by

$$a \cdot (f_1 \wedge f_2 \wedge \cdots f_m) := (a \cdot f_1) \wedge \cdots \wedge (a \cdot f_m)$$

where f_j are polynomials. We define the infinitesimal action of $\xi \in \mathfrak{sp}(2n,\mathbb{R})$ by

$$\xi \cdot (f_1 \wedge f_2 \wedge \cdots f_m) := \frac{d}{dt} (\exp(t\xi) \cdot (f_1 \wedge f_2 \wedge \cdots f_m))_{|t=0} \ .$$

Remark 3.4 $f_1 \wedge f_2 \wedge \cdots f_m$ looks strange, but the correponding real object is $\mathcal{H}_{-f_1} \wedge \mathcal{H}_{-f_2} \wedge \cdots \wedge \mathcal{H}_{-f_m}$. Thus the degree of f_j is 1, and so $f \wedge g = -g \wedge f$ for each functions f, g.

Proposition 3.1 The above infinitesimal action by $\xi \in \mathfrak{sp}(2n,\mathbb{R})$ is a derivation of degree 0 and

$$\xi \cdot f = \{\hat{J}(\xi), f\} \quad \text{for each polynomial } f \ .$$

Proof: The first assersion of being derivation of degree 0 is trivial. The second assersion is:

$$\xi \cdot f := \frac{d}{dt}\exp(t\xi)\cdot f_{|t=0} = \frac{d}{dt} f \circ \exp(-t\xi)_{|t=0} = \langle df, -\xi_M\rangle = \omega(\mathcal{H}_f, -\mathcal{H}_{\hat{J}(\xi)}) = \{\hat{J}(\xi), f\}.$$

This corresponds to $\xi \cdot \mathcal{H}_f := \dfrac{d}{dt}\exp(t\xi)\cdot \mathcal{H}_{f|t=0} = \mathcal{L}_{-\xi_M}\mathcal{H}_f = -[\mathcal{H}_{\hat{J}(\xi)}, \mathcal{H}_f] = \mathcal{H}_{\{\hat{J}(\xi),f\}}$. ∎

To determine relative cochain complex $C^\bullet(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))$, there are two conditions, one is $i_\xi \sigma = 0$, and the other is $i_\xi d\,\sigma = 0$ for each m-cochain σ, where i_ξ is the interior product with respect to $\xi \in \mathfrak{k} = \mathfrak{sp}(2n,\mathbb{R}) = S^2$. Since i_ξ is a skew-derivation of degree -1, in order to know the effect of i_ξ, it is enough to know the operation of $i_\xi\sigma$ for 1-cochain σ. Going back to Hamiltonian vector fields, we see that

$$i_\xi \sigma = \langle \sigma, \xi_M\rangle = \langle \sigma, \mathcal{H}_{\hat{J}(\xi)}\rangle = \langle \sigma, -\hat{J}(\xi)\rangle.$$

In general, it holds

$$(i_\xi\sigma)(f_1, f_2, \ldots) = \sigma(-\hat{J}(\xi), f_1, f_2, \ldots) \quad \text{for} \quad \sigma \in C^m(\mathfrak{ham}^0_{2n}).$$

Proposition 3.2 Let $\sigma \neq 0$ and $\sigma \in \Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_3}\mathfrak{S}_3 \otimes \cdots$ (where $\sum k_j = m$). If $i_\xi\sigma = 0$ for $\forall \xi \in \mathfrak{k} = \mathfrak{sp}(2n,\mathbb{R})$, then $\sigma \in \Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots$.

Proof: Since $i_\xi\sigma = -\langle \sigma, \hat{J}(\xi)\rangle$ for 1-cochain and $\hat{J}(\xi) \in S^2$, we see that $i_\xi\sigma = 0$ if $\sigma \in \mathfrak{S}_\ell$ with $\ell \neq 2$. If $\sigma \in \mathfrak{S}_2$ satisfies $i_\xi\sigma = 0$ for $\forall\xi \in \mathfrak{sp}(2n,\mathbb{R})$, then $\sigma = 0 \in \mathfrak{S}_2$ because $S^2 = \mathfrak{sp}(2n,\mathbb{R})$ is semi-simple and $\hat{J}(\xi)$ generate S^2. Now we may rewrite

$$\sigma = \sum_A \tau_A \wedge \rho_A$$

where $\tau_A \in \Lambda^{k_2}\mathfrak{S}_2$, $\rho_A \in \Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots$ and ρ_A are linearly independent. Using the fact $i_\xi \rho_A = 0$ for $\forall \xi \in \mathfrak{k}$, we have $0 = i_\xi\sigma = \sum_A i_\xi\tau_A \otimes \rho_A$ and those imply that $i_\xi\tau_A = 0$ for $\forall\xi \in \mathfrak{sp}(2n,\mathbb{R})$ and A. If $0 < k_2 \leq \dim\mathfrak{S}_2 = n(2n+1)$ then $\tau_A = 0$ for $\forall A$ if necessary we can use $i_{\xi_{k_2}} \cdots i_{\xi_1}\tau_A = 0$. Thus $\sigma = 0$. ∎

Thus, the condition $i_\xi\sigma = 0$ for $\xi \in \mathfrak{sp}(2n,\mathbb{R})$ imply that $\mathfrak{S}_2$ does not appear in $C^{\bullet}_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))$ and

$$\begin{aligned} C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R})) &= \Lambda^m(\mathfrak{S}_3 \oplus \cdots)^{Sp(2n,\mathbb{R})} \\ &= \sum_{\sum k_j = m} \left(\Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots\right)^{Sp(2n,\mathbb{R})} \end{aligned}$$

and so

$$C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w = \sum_{\substack{\sum k_j = m,\\ \sum (j-2)k_j = w}} \left(\Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots\right)^{Sp(2n,\mathbb{R})}$$

The other condition of being relative cochain is $i_\xi d\,\sigma = 0$ for each m-cochain σ. Again going back to Hamiltonian vector fields, for each 1-cochain σ, we see that

$$\langle i_\xi d\,\sigma, \mathcal{H}_f\rangle = (d\,\sigma)(\xi_M, \mathcal{H}_f) = -\langle \sigma, [\xi_M, \mathcal{H}_f]\rangle = -\langle \sigma, [\mathcal{H}_{\hat{J}(\xi)}, \mathcal{H}_f]\rangle = \langle \sigma, \mathcal{H}_{\{\hat{J}(\xi), f\}}\rangle$$

and so

$$\langle i_\xi d\,\sigma, f\rangle = \langle \sigma, \{\hat{J}(\xi), f\}\rangle = \langle \sigma, \xi \cdot f\rangle$$

for 1-cochain σ, and $\xi \in \mathfrak{sp}(2n,\mathbb{R}), f \in C^\infty(M)$.
For m-cochain $\tau = \sigma_1 \wedge \cdots \wedge \sigma_m$ (σ_j are 1-cochains), since d is a skew-derivation of degree $+1$, we have

$$d\,\tau = \sum_{j=1}^m (-1)^{j+1}\sigma_1 \wedge \cdots \wedge d\,\sigma_j \wedge \cdots \wedge \sigma_m,$$

and since the interior product i_ξ for each $\xi \in \mathfrak{sp}(2n,\mathbb{R})$ is a skew-derivation of degree -1, we have

$$\begin{aligned} (i_\xi \circ d\,)\,\tau =& i_\xi \sum_j (-1)^{j+1}\sigma_1 \wedge \cdots \wedge d\,\sigma_j \wedge \cdots \wedge \sigma_m \\ =& \sum_{\ell<j} (-1)^{j+1}(-1)^{\ell+1}\sigma_1 \wedge \cdots \wedge i_\xi\sigma_\ell \wedge \cdots \wedge d\,\sigma_j \wedge \cdots \wedge \sigma_m \\ &+ \sum_j (-1)^{j+1}(-1)^{j+1}\sigma_1 \wedge \cdots \wedge ((i_\xi \circ d\,)\sigma_j) \wedge \cdots \wedge \sigma_m \\ &+ \sum_{\ell>j} (-1)^{j+1}(-1)^{\ell+2}\sigma_1 \wedge \cdots \wedge d\,\sigma_j \wedge \cdots \wedge i_\xi\sigma_\ell \wedge \cdots \wedge \sigma_m \end{aligned}$$

If $i_\xi\sigma_j = 0$ $(j = 1,\ldots,m)$, then we have

$$(i_\xi \circ d\,)\,\tau = \sum_j \sigma_1 \wedge \cdots \wedge ((i_\xi \circ d\,)\sigma_j) \wedge \cdots \wedge \sigma_m,$$

namely, $i_\xi \circ d$ is a derivation of degree 0. Thus, under the condition of $i_\xi\sigma = 0$ for any cochain σ and $\xi \in \mathfrak{sp}(2n,\mathbb{R})$, $i_\xi \circ d$ becomes an ordinary derivation of degree 0.

Proposition 3.3 Let σ be a m-cochain with $i_\xi\sigma = 0$ for $\forall\xi \in \mathfrak{sp}(2n,\mathbb{R})$. Then $(i_\xi \circ d\,)$ behaves as derivation of degree 0 and characterized by $\langle (i_\xi \circ d\,)\sigma, f\rangle = \langle \sigma, \{\hat{J}(\xi), f\}\rangle$ for each 1-cochain σ and $\xi \in \mathfrak{sp}(2n,\mathbb{R})$.

Remark 3.5 It may be a better way to recall Cartan's formula $\mathcal{L}_\xi = d \circ i_\xi + i_\xi \circ d$ in order to prove the above.

The group $Sp(2n,\mathbb{R})$ acts on cochain complexes as the dual acton of that of on the exterior algebra of polynomial functions on $\mathbb{R}^{2n}$. Thus, the precise definition is

$$\begin{aligned} (a \cdot \sigma)\,(f_1,\ldots,f_m) &= \sigma\left(a^{-1}\cdot f_1,\ldots,a^{-1}\cdot f_m\right) = \sigma\left(f_1 \circ a_M,\ldots,f_m \circ a_M\right) \\ \langle a \cdot \sigma, f_1 \wedge \ldots \wedge f_m\rangle &= \langle \sigma, a^{-1}\cdot(f_1 \wedge \ldots \wedge f_m)\rangle = \langle \sigma, f_1 \circ a_M \wedge \cdots \wedge f_m \circ a_M\rangle \end{aligned}$$

for a general m-cochian σ.

Remark 3.6 We re-confirm here that our coboundary operator d is compatible with the group-action of $Sp(2n,\mathbb{R})$, i.e,. $a\cdot d = d\circ a_M$ holds for each symplectic automorphism a. It is enough only to show for 1-cochain σ. We see that

$$\begin{aligned}(d\,(a\cdot\sigma))\,(f,g) &= -\langle a\cdot\sigma,\{f,g\}\rangle = -\langle\sigma,\{f,g\}\circ a_M^{-1}\rangle = -\langle\sigma,\{f\circ a_M^{-1}, g\circ a_M^{-1}\}\rangle\\ &= (d\,\sigma)(f\circ a_M^{-1}, g\circ a_M^{-1}) = (d\,\sigma)(a\cdot f, a\cdot g).\end{aligned}$$

for $\forall a\in Sp(2n,\mathbb{R})$.

Definition 3 We define the infinitesimal action of $\xi\in\mathfrak{sp}(2n,\mathbb{R})$ for each cochain σ by

$$\xi\cdot\sigma = \frac{d}{dt}\exp(t\xi)\cdot\sigma_{|t=0}\ .$$

Proposition 3.4 The infinitesimal action $\xi\in^(2n,\mathbb{R})$ preserves the cochain complex $C^m(\mathfrak{ham}^0_{2n})$, behaves as an ordinary derivation of degree 0 and

$$\langle\xi\cdot\sigma,f\rangle = -\langle\sigma,\xi\cdot f\rangle = -\langle\sigma,\{\hat{J}(\xi),f\}\rangle = -\langle(i_\xi d\,\sigma,f\rangle\ .$$

Thus, $(i_\xi d)(\sigma) = -\xi\cdot\sigma$ holds for each m-cochain σ with $i_\xi\sigma = 0$ ($\xi\in\mathfrak{sp}(2n,\mathbb{R})$).

An advantage of the relation in Proposition 3.4 is that $\xi : C^m(\mathfrak{ham}^0_{2n}) \longrightarrow C^m(\mathfrak{ham}^0_{2n})$ is a derivation of degree 0 with respect to the wedge product but also a derivation of degree 0 inside of 1-cochain. Namely, each 1-cochain is a linear combination of symmetric powers and since $f\mapsto\{\hat{J}(\xi),f\}$ is a deravation for each polynomial function f, we have

$$\begin{aligned}\xi\cdot\left({e_1}^{k_1}\cdots{e_\ell}^{k_\ell}\cdots{e_{2n}}^{k_{2n}}\right) &= \sum_\ell {e_1}^{k_1}\cdots\xi\cdot\left({e_\ell}^{k_\ell}\right)\cdots{e_{2n}}^{k_{2n}}\\ &= \sum_\ell {e_1}^{k_1}\cdots\cdot\left(k_\ell {e_\ell}^{k_\ell-1}\xi(e_\ell)\right)\cdots{e_{2n}}^{k_{2n}}\ .\end{aligned}$$

4 Decomposition of m-th cochain complex of weight w

Our concern in this section is, given a pair of positive integers (m,w), find all possibilities of cochain complex by denoting the sequences $(k_3,k_4,\ldots)$ of non-negative integers of multiplicity which satisfies $\sum_{j\geq 3} k_j = m$ and $\sum_{j\geq 3}(j-2)k_j = w$. We have to be careful about $\Lambda^{k_\ell}\mathfrak{S}_\ell = \{0\}$ may be happen when $k_{ell} > \dim\mathfrak{S}_\ell$, where $\dim\mathfrak{S}_\ell = (\ell+2n-1)!/(\ell!(2n-1)!)$.
By shifting the indices by -2, we rearrange our situation as below. Given a pair of non-negative integers (m,w), we would like to find all sequences $(\hat{k}_1,\hat{k}_2,\ldots)$ of non-negative integers satisfying

$$\sum_{j\geq 1}\hat{k}_j = m \quad\text{and}\quad \sum_{j\geq 1} j\hat{k}_j = w \tag{1}$$

$m\leq w$ is a necessary condition and denote the set of sequences $(\hat{k}_1,\hat{k}_2,\ldots)$ satisfying (1) by $F(m,w)$, By subtracting the first equation of (1) from the second one of (1), we have

$$\sum_{j\geq 2}(j-1)\hat{k}_j = w-m\ .$$

Thus, we have the all solutions of (1), $F(m,w)$ by the next recursive formula:

$$F(m,w) = \bigsqcup_{\hat{k}_1=\max(0,2m-w)}^{m} \{(\hat{k}_1,x)\mid x\in F(m-\hat{k}_1,w-m)\}$$

For a given (m, w), we have the Maple script `gkf_act-1.mpl`, which shows us the all solutions of (1), There is a primitive question "do the solutions exist for all $m \leq w$? or how many?". We will give an answer to this question, here. We join our two equations into one equation as below:

$$\begin{aligned} w &= \hat{k}_1 + 2\hat{k}_2 + \cdots + s\hat{k}_s \\ &= (\underbrace{1 + \cdots + 1}_{\hat{k}_1}) + (\underbrace{2 + \cdots + 2}_{\hat{k}_2}) + \cdots + (\underbrace{s + \cdots + s}_{\hat{k}_s}) \\ &= (\underbrace{s + \cdots + s}_{\hat{k}_s}) + \cdots + (\underbrace{2 + \cdots + 2}_{\hat{k}_2}) + (\underbrace{1 + \cdots + 1}_{\hat{k}_1}) \\ &= \ell_1 + \ell_2 + \cdots + \ell_m \end{aligned}$$

where $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 1$. Thit is a partition of w with length m or a Young diagram of height m with w cells. Conversely, for a partition of w

$$\begin{aligned} w &= \ell_1 + \ell_2 + \cdots + \ell_m \\ \ell_1 &\geqq \ell_2 \geqq \cdots \geqq \ell_m \geqq 1 \end{aligned} \tag{2}$$

$\hat{k}_i := \#\{j \mid \ell_j = i\}$ gives a solution of (1). That means there is a one-to-one correspondence between the solution of (1) and all the partitions of w with length m or a Young diagram of height m with w cells.

Remark 4.1 Be careful of difference of definition of $\hat{k}_i := \#\{j \mid \ell_j = i\}$ and the conjugate Young diagram $c_i := \#\{j \mid \ell_j \geqq i\}$.

Proposition 4.1 By $r(m, w)$ we mean the number of solutions of (1).

For integers $m > 0$, $w \geqq 0$, we define $\tilde{r}(m, w)$ the number of solutions of

$$\begin{aligned} w &= \ell_1 + \ell_2 + \cdots + \ell_m \\ \ell_1 &\geqq \ell_2 \geqq \cdots \geqq \ell_m \geqq 0 \end{aligned} \tag{3}$$

By an elementary observation below

$$w = \ell_1+\ell_2+\cdots+\ell_m \quad \text{with} \quad \ell_i \geq 0 \quad \text{and} \quad w+m = (\ell_1+1)+(\ell_2+1)+\cdots+(\ell_m+1) \quad \text{with} \quad (\ell_i+1) > 0$$

we see $\tilde{r}(m, w) = r(m, w + m)$. The generating function of $\tilde{r}(m, w)$ is known as

$$\sum_{c=0}^{\infty} \tilde{r}(m, c)x^c = \prod_{k=1}^{m} \left(\frac{1}{1 - x^k}\right).$$

Example 4.1 Let us try to know all possibilities when weight=2 case.
When $m = 1$, then $\ell_1 = 2$, so we have $\hat{k}_2 = 1$ and $\hat{k}_j = 0$ $(j \neq 2)$. Thus, $k_4 = 1$.
When $m = 2$, then $2 = \ell_1 + \ell_2$ ($\ell_1 \geq \ell_2 \geq 1$), $\ell_1 = \ell_2 = 1$, so we have $\hat{k}_1 = 2$ and $\hat{k}_j = 0$ $(j \neq 1)$. Thus, $k_3 = 2$. These show

$$\begin{aligned} &C^1_{GF}(\mathfrak{ham}^0_{2n})_2 = \mathfrak{S}_4 && C^1_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_2 = \mathfrak{S}_4^{Sp(2n,\mathbb{R})} (= \{\mathbf{0}\}) \\ &C^2_{GF}(\mathfrak{ham}^0_{2n})_2 = \Lambda^2\mathfrak{S}_3 && C^2_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_2 = (\Lambda^2\mathfrak{S}_3)^{Sp(2n,\mathbb{R})} \end{aligned}$$

In the above, $\mathfrak{S}_4$ is an irreducible representation of $Sp(2n, \mathbb{R})$ and so $\mathfrak{S}_4^{Sp(2n,\mathbb{R})} = \{\mathbf{0}\}$. Also, concerning $\Lambda^2\mathfrak{S}_3$, if n=1 we know $\Lambda^2\mathfrak{S}_3 = \mathfrak{S}_0 \oplus \mathfrak{S}_4$ as we will see in Example 5.1. If n=2 then by the help of Littlewood-Richardson rule, we get a little complex expression $\Lambda^2\mathfrak{S}_3 = \mathfrak{S}_0 \oplus \mathfrak{S}_4 \oplus V_{<(1,1)>} \oplus V_{<(2,2)>} \oplus V_{<(3,3)>} \oplus V_{<(5,1)>}$, where $V_{<(p,q)>}$ is the irreducible representation of the natural action of $Sp(4, \mathbb{R})$ on $\mathbb{R}^4$, corresponding the Young diagram (p, q), and $V_{<(p,0)>} = \mathfrak{S}_p$. Thus, if n=1 or 2, then we see $\dim C^2_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_2 = 1$. Actually, we get a lot of help from representation theory in this project.

Example 4.2 weight=4 case:
When $m = 2$, i.e., $4 = \ell_1+\ell_2$ ($\ell_1 \geq \ell_2 \geq 1$), then $(\ell_1, \ell_2) = (3, 1)$ or $(2, 2)$, so we have $(\hat{k}_1 = 1, \hat{k}_3 = 1)$, or $\hat{k}_2 = 2$. Thus, $(k_3 = 1, k_5 = 1)$ or $(k_4 = 2)$. When $m = 3$, i.e., $4 = \ell_1 + \ell_2 + \ell_3$ ($\ell_1 \geq \ell_2 \geq \ell_3 \geq 1$), $\ell_1 = 2, \ell_2 = 1, \ell_3 = 1$. Thus $(\hat{k}_2 = 1, \hat{k}_1 = 2)$, so $(k_3 = 2, k_4 = 1)$.
These show

$$
\begin{aligned}
&C^1_{GF}(\mathfrak{ham}^0_{2n})_4 = \mathfrak{S}_6 && C^2_{GF}(\mathfrak{ham}^0_{2n})_4 = (\mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus \Lambda^2\mathfrak{S}_4\\
&C^3_{GF}(\mathfrak{ham}^0_{2n})_4 = \Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_4 && C^4_{GF}(\mathfrak{ham}^0_{2n})_4 = \Lambda^4\mathfrak{S}_3
\end{aligned}
$$

Example 4.3 By the same way, we have weight=6 case:

$$
\begin{aligned}
C^1_{GF}(\mathfrak{ham}^0_{2n})_6 &= \mathfrak{S}_8\\
C^2_{GF}(\mathfrak{ham}^0_{2n})_6 &= (\mathfrak{S}_3 \otimes \mathfrak{S}_7) \oplus (\mathfrak{S}_4 \otimes \mathfrak{S}_6) \oplus \Lambda^2\mathfrak{S}_5\\
C^3_{GF}(\mathfrak{ham}^0_{2n})_6 &= \left(\Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_6\right) \oplus (\mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5) \oplus \Lambda^3\mathfrak{S}_4\\
C^4_{GF}(\mathfrak{ham}^0_{2n})_6 &= \left(\Lambda^3\mathfrak{S}_3 \otimes \mathfrak{S}_5\right) \oplus \left(\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4\right)\\
C^5_{GF}(\mathfrak{ham}^0_{2n})_6 &= \Lambda^4\mathfrak{S}_3 \otimes \mathfrak{S}_4\\
C^6_{GF}(\mathfrak{ham}^0_{2n})_6 &= \Lambda^6\mathfrak{S}_3
\end{aligned}
$$

If $n = 1$, then $\dim\mathfrak{S}_3 = 4$ and so we have $C^6_{GF}(\mathfrak{ham}^0_2)_6 = \{\mathbf{0}\}$.

We express the data which we got by the next table in short form, dividing into direct sum components. Our abbreviation rule is that

1. only pick up the i's with $k_i > 0$,
2. if $k_i > 1$ then express the multiplicity by the power like i^{k_i}, i.e., $\left(i^{k_i}\ j^{k_j}\ \cdots\right)$
3. if $k_i = 1$ then only write i

Using the rule, Example 4.3 above with weight=6 can be written in the next table:

degree	ref.#	type
1	1	(8)
2	1	(3 7)
	2	(4 6)
	3	(5^2)

degree	ref.#	type
3	1	(3^2 6)
	2	(3 4 5)
	3	(4^3)

degree	ref.#	type
4	1	(3^3 5)
	2	(3^2 4^2)
5	1	(3^4 4)
6	1	(3^6)

5 Aid from Representation theory of $Sp(2n, \mathbb{R})$

In order to compute the relative cohomology groups, we have to know some basis of the cochain complex and the concrete matrix representation of the coboundary operator, and that rank. Fortunately, in the case of $C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w$, there is a very sophisticated way to know the dimension without knowing basis. Namely, wan consider the natural representation of $Sp(2n, \mathbb{R})$ acting on $\mathbb{R}^{2n}$. Then the all irreducible representations are known by Young diagram of length at most n. Also, the p-th symmetric tensor product of $\mathbb{R}^{2n}$ is an irreducible representation and is identified with $\mathfrak{S}_p$ or its dual in this paper. The corresponding Young diagram is $(p) = \underbrace{\square\square\cdots\square\square}_{p-\text{times}}$.

Since

$$
C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n, \mathbb{R}))_w = \sum_{\sum k_j = m,\ \sum (j-2)k_j = w} \left(\Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots\right)^{Sp(2n,\mathbb{R})}
$$

if we know the index of the trivial representation in

$$\sum_{\sum k_j = m,\ \sum (j-2)k_j = w} \left(\Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots \right)$$

by some help of representation theory of $Sp(2n,\mathbb{R})$, that is just equal to $\dim C^m_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$. Thus, we can calculate the Euler characteristic number without knowing cohomology data at least theoretically. When $n = 1$, the representation theory is rather clear and that is very helpful as we see later.

5.1 Assistance of $Sp(2,\mathbb{R})$

Instead of the variables (q^1, p_1) in $\mathbb{R}^2$, we use the classical notation (x, y), and let $\widehat{z}^a_k$ be $\dfrac{x^a}{a!}\dfrac{y^{k-a}}{(k-a)!} \in S^k$ and the dual basis in $\mathfrak{S}_k$ of $\widehat{z}^a_k$ by the notation z^a_k. Now, the Poisson bracket $\{f, g\}$ is the Jacobian $\dfrac{\partial(f,g)}{\partial(x,y)}$ and get the relations $\{xy, x^2\} = -2x^2$, $\{xy, y^2\} = 2y^2$, $\{x^2, y^2\} = 4xy$ and so we have the correspondence below with the famous matrices through the momentum mapping J of the natural symplectic action $Sp(2,\mathbb{R})$:

$$xy \leftrightarrow H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{y^2}{2} \leftrightarrow X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, -\frac{x^2}{2} \leftrightarrow Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and we have already seen that the component S^2 is a subalgebra of $\mathfrak{ham}_2$ and isomorphic to $\mathfrak{sp}(\mathbb{R}^2, \omega) \cong \mathfrak{sl}(2,\mathbb{R})$.
It is well-known that S^ℓ or the (symplectic) dual space $\mathfrak{S}_\ell$ are the irreducible representations of $Sp(2,\mathbb{R})$ and some useful formulae, for instance, an irreducible decomposition of tensor product is the next:

$$\mathfrak{S}_k \otimes \mathfrak{S}_\ell = \mathfrak{S}_{k+\ell} \oplus \mathfrak{S}_{k+\ell-2} \oplus \cdots \oplus \mathfrak{S}_{|k-\ell|}$$

Since $C^m_{GF}(\mathfrak{ham}^0_2)_w = \displaystyle\sum_{\sum k_j = m,\ \sum (j-2)k_j = w} \left(\Lambda^{k_2}\mathfrak{S}_2 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots \right)$, if we know the irreducible decomposition of $\Lambda^k\mathfrak{S}_\ell$, then after tensor product-ing those, we have the complete irreducible decomposition, and can pick up the trivial representation,

Example 5.1 As shown in Example 4.1, $\Lambda^2\mathfrak{S}_3$ is a component of $C^2_{GF}(\mathfrak{ham}^0_{2n})_2$, we decompose $\Lambda^2\mathfrak{S}_3$ into irreducible components when $n = 1$. $z^{\ell_1}_3 \wedge z^{\ell_2}_3$ $(0 \le \ell_1 < \ell_2 \le 3)$ are a basis of $\Lambda^2\mathfrak{S}_3$. We will find the weight vector space from

$$T = \sum_{0 \le \ell_1 < \ell_2 < 3} c_{\ell_1,\ell_2} z^{\ell_1}_3 \wedge z^{\ell_2}_3$$

which must be zero by the dual action of X. Since our action is $z^0_3 \mapsto 3z^1_3$, $z^1_3 \mapsto 2z^2_3$, $z^2_3 \mapsto 1z^3_3$, and $z^3_3 \mapsto 0$, we have

$$\begin{aligned} 0 =& c_{01}(3z^1_3 \wedge z^1_3 + z^0_3 \wedge 2z^2_3) + c_{02}(3z^1_3 \wedge z^2_3 + z^0_3 \wedge z^3_3) + c_{03}(3z^1_3 \wedge z^3_3 + 0) \\ &+ c_{12}(2z^2_3 \wedge z^2_3 + z^1_3 \wedge z^3_3) + c_{13}(2z^2_3 \wedge z^3_3 + 0) + c_{23}(z^3_3 \wedge z^3_3 + 0) \\ =& 2c_{01} z^0_3 \wedge z^2_3 + c_{02} z^0_3 \wedge z^3_3 + 3c_{02} z^1_3 \wedge z^2_3 + (3c_{03} + c_{12}) z^1_3 \wedge z^3_3 + 2c_{13} z^2_3 \wedge z^3_3 \end{aligned}$$

By solving a homogeneous linear equations, we get $T = c_{03}(z^0_3 \wedge z^3_3 - 3z^1_3 \wedge z^2_3) + c_{23} z^2_3 \wedge z^3_3$, and $z^0_3 \wedge z^3_3 - 3z^1_3 \wedge z^2_3$ is the highest weight vector of $\mathfrak{S}_0$ and $z^2_3 \wedge z^3_3$ is of $\mathfrak{S}_4$, and we have $\Lambda^2\mathfrak{S}_3 = \mathfrak{S}_0 \oplus \mathfrak{S}_4$. Thus, when $n = 1$ and the weight=2 case, we add our decomposition and we see

$$\begin{aligned} &C^1_{GF}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_2 = \mathfrak{S}_4^{Sp(2,\mathbb{R})} = \{\mathbf{0}\} \\ &C^2_{GF}(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_2 = (\Lambda^2\mathfrak{S}_3)^{Sp(2,\mathbb{R})} = (\mathfrak{S}_0 \oplus \mathfrak{S}_4)^{Sp(2,\mathbb{R})} = \mathfrak{S}_0 = \mathbb{R} \end{aligned}$$

since $\mathfrak{S}_q^{Sp(2n,\mathbb{R})} = \{\mathbf{0}\}$ for $q > 2$.

Example 5.2 $\Lambda^2\mathfrak{S}_4$ is a component of $C^2_{GF}(\mathfrak{ham}^0_{2n})_4$ in Example 4.2. When $n=1$, we can decompose it into the irreducible components as the same way above, and get $\Lambda^2\mathfrak{S}_4 = \mathfrak{S}_2 \oplus \mathfrak{S}_6$.
When $n=1$, since we have the next decomposition $\Lambda^3\mathfrak{S}_5 \cong \mathfrak{S}_9 \oplus \mathfrak{S}_5 \oplus \mathfrak{S}_3$, we have no rule of negative 4-step descending nor 2-step descending for $\Lambda^p\mathfrak{S}_q$.

We would like to see the effect of tensor product.

Example 5.3 When weight=6, we know that $C^2_{GF}(\mathfrak{ham}^0_{2n})_6$ has 3 components $(3,7)$, $(4,6)$, (5^2), and $C^3_{GF}(\mathfrak{ham}^0_{2n})_6$ has 3 components $(3^2,6)$, $(3,4,5)$, (4^3), and $C^4_{GF}(\mathfrak{ham}^0_{2n})_6$ has 2 components $(3^3,5)$, $(3^2,4^2)$, and $C^5_{GF}(\mathfrak{ham}^0_{2n})_6$ has 1 component $(3^4,4)$, and $C^6_{GF}(\mathfrak{ham}^0_{2n})_6$ has 1 component (3^6).
Assume $n=1$. Concerning with degree 2 cases, since $\mathfrak{S}_3 \otimes \mathfrak{S}_7 = \mathfrak{S}_4 \oplus \cdots \oplus \mathfrak{S}_{10}$ and $\mathfrak{S}_4 \otimes \mathfrak{S}_6 = \mathfrak{S}_2 \oplus \cdots \oplus \mathfrak{S}_{10}$, we see that

$$[\mathfrak{S}_3 \otimes \mathfrak{S}_7, \mathfrak{S}_0] = 0 \quad \text{and} \quad [\mathfrak{S}_4 \otimes \mathfrak{S}_6, \mathfrak{S}_0] = 0$$

On the other hand, we get $\Lambda^2\mathfrak{S}_5 = \mathfrak{S}_0 \oplus \mathfrak{S}_4 \oplus \mathfrak{S}_8$, we have

$$[\Lambda^2\mathfrak{S}_5, \mathfrak{S}_0] = 1$$

degree 3: (3.1)-case: $\Lambda^2(\mathfrak{S}_3) \otimes \mathfrak{S}_6 = (\mathfrak{S}_0 \oplus \mathfrak{S}_4) \otimes \mathfrak{S}_6 = \mathfrak{S}_6 + (\mathfrak{S}_2 + \cdots + \mathfrak{S}_{10})$, and so

$$[\Lambda^2(\mathfrak{S}_3) \otimes \mathfrak{S}_6, \mathfrak{S}_0] = 0$$

$$\begin{aligned}\text{(3.2)-case:}\quad \mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5 =& (\mathfrak{S}_1 + \mathfrak{S}_3 + \mathfrak{S}_5 + \mathfrak{S}_7) \otimes \mathfrak{S}_5 \\ =& (\mathfrak{S}_4 + \mathfrak{S}_6) + (\mathfrak{S}_2 + \mathfrak{S}_4 + \mathfrak{S}_6 + \mathfrak{S}_8) + (\mathfrak{S}_0 + \mathfrak{S}_2 + \mathfrak{S}_4 + \mathfrak{S}_6 + \mathfrak{S}_8 + \mathfrak{S}_{10}) \\ &+ (\mathfrak{S}_2 + \mathfrak{S}_4 + \mathfrak{S}_6 + \mathfrak{S}_8 + \mathfrak{S}_{10} + \mathfrak{S}_{12})\end{aligned}$$

we see that

$$[\mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5, \mathfrak{S}_0] = 1$$

Since $\Lambda^3\mathfrak{S}_4 = \mathfrak{S}_2 + \mathfrak{S}_6$,

$$[\Lambda^3\mathfrak{S}_4, \mathfrak{S}_0] = 0$$

degree 4: (4.1)-case: $(\Lambda^3\mathfrak{S}_3) \otimes \mathfrak{S}_5 = \mathfrak{S}_3 \otimes \mathfrak{S}_5 = \mathfrak{S}_2 + \cdots + \mathfrak{S}_8$, here we used the property that $\Lambda^k W = \Lambda^{\dim W - k} W$ in general, and

$$[(\Lambda^3\mathfrak{S}_3) \otimes \mathfrak{S}_5, \mathfrak{S}_0] = 0$$

(4.2)-case: $(\Lambda^2\mathfrak{S}_3) \otimes (\Lambda^2\mathfrak{S}_4) = (\mathfrak{S}_0 + \mathfrak{S}_4) \otimes (\mathfrak{S}_2 + \mathfrak{S}_6) = (\mathfrak{S}_2 + \mathfrak{S}_6) + (\mathfrak{S}_2 + \mathfrak{S}_4 + \mathfrak{S}_6) + (\mathfrak{S}_2 + \cdots + \mathfrak{S}_{10})$,

$$[(\Lambda^2\mathfrak{S}_3) \otimes (\Lambda^2\mathfrak{S}_4), \mathfrak{S}_0] = 0$$

(5)-case: $(\Lambda^4\mathfrak{S}_3) \otimes \mathfrak{S}_4 = \mathfrak{S}_0 \otimes \mathfrak{S}_4 = \mathfrak{S}_4$ because of $\dim\mathfrak{S}_3 = 4$,

$$[(\Lambda^4\mathfrak{S}_3) \otimes \mathfrak{S}_4, \mathfrak{S}_0] = 0$$

We add those facts in the table in Example 4.3.

degree	ref.#	type	dim
1	1	(8)	0
2	1	(3 7)	0
	2	(4 6)	0
	3	(5^2)	1

degree	ref.#	type	dim
3	1	$(3^2\ 6)$	0
	2	(3 4 5)	1
	3	(4^3)	0

degree	ref.#	type	dim
4	1	$(3^3\ 5)$	0
	2	$(3^2\ 4^2)$	0
5	1	$(3^4\ 4)$	0
6	1	(3^6)	0

We summarize the ideas, so far. For a given weight w and degree m, we have the complete list of subcomplex of type $(k_3, k_3, \ldots, k_s)$ such that $\displaystyle\sum_j (j-2)k_j = w$ and $\displaystyle\sum_j k_j = w$ and

$$C^m(\mathfrak{ham}^0_2)_w = \oplus \Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots \Lambda^{k_s}\mathfrak{S}_s$$

It is possible to decompose $\Lambda^p\mathfrak{S}_q$ into irreducible subspaces, say, $\Lambda^p\mathfrak{S}_q = \sum \alpha^p_{qr}\mathfrak{S}_r$, where $\alpha^p_{qr} \in \mathbb{Z}_{\geq 0}$. (But, it is not clear if those α^p_{qr} have some "rule".) Since we have the tensor product formula

$$\mathfrak{S}_p \otimes \mathfrak{S}_q = \mathfrak{S}_{|p-q|} \oplus \mathfrak{S}_{|p-q|+2} \oplus \cdots \mathfrak{S}_{p+q}$$

we can decompose $\Lambda^{k_3}\mathfrak{S}_3 \otimes \Lambda^{k_4}\mathfrak{S}_4 \otimes \cdots \Lambda^{k_s}\mathfrak{S}_s$ into irreducible components, thus we can divide $C^m(\mathfrak{ham}^0_2)_w$ into irreducible components, and therefore we can pick up the multiplicity of the trivial representation, and we know the dimension of $C^m(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_w$ as that multiplicity.

Theorem 1 There is a sequence of computer programs which follow the mathematical story above. If we input the weight w, then we can get $\dim C^m(\mathfrak{ham}^0_2, \mathfrak{sp}(2,\mathbb{R}))_w$ and the precise contributions of subcomplex, and the Euler characteristic number. If w is big, we may face some trouble of shortage of memory and so on. Nevertheless, so for we have a table of dimensions of relative cochain complex until $w \leq 20$. The table below means the horizontal direction is the degree of cochain complex and the descending vertical direction means increasing weight.

w	1	2	3	4	5	6	7	8	9	10	11	12	13	14	χ
2	0	1													2
4	0	0	1	1											1
6	0	1	1	0	0										1
8	0	0	4	5	1	0									1
10	0	1	3	9	12	4	0								0
12	0	0	8	23	22	13	5	0							2
14	0	1	6	31	71	58	15	2	1						0
16	0	0	12	61	126	147	95	24	0						0
18	0	1	10	80	262	380	268	100	21	1					2
20	0	0	17	124	423	791	801	414	96	9	1				1
22	0	1	14	163	738	1586	1874	1276	479	82	3				1
24	0	0	23	229	1091	2897	4281	3534	1628	408	49	1			-2
26	0	1	19	285	1722	5102	8613	8735	5222	1703	266	19	1		3
28	0	0	29	385	2428	8465	16905	19930	14133	5981	1408	144	2		1
30	0	1	25	468	3541	13661	30687	42291	35986	18457	5431	855	63	0	1

Here, χ means the alternating sum of the dimension of relative cochain complex, including 0-dimensional cochain complex $\mathbb{R}$. Thus, $\chi \neq 1$ means there is non-trivial Betti number.

Remark 5.1 By help of symbol calculus, we get more informations of the Euler characteristics: $\chi(32) = 2$, $\chi(34) = -2$, $\chi(36) = 4$, $\chi(38) = -5$, $\chi(40) = 5$, $\chi(42) = 2$, $\chi(44) = 2$, $\chi(46) = 3$, $\chi(48) = 0$, $\chi(50) = 1$.
In [3], they express the generating function as

$$\sum_{w=0}^{\infty} \chi(\mathrm{H}^*_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{s}p(2,\mathbb{R}))_w t^w = 1 + t^2 - t^{10} + t^{12} - t^{14} - t^{16} + t^{18} - 3t^{24} + 2t^{26} + \cdots$$

Adding our results, we may write

$$\sum_{w=0}^{\infty} \chi(\mathrm{H}^*_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{s}p(2,\mathbb{R}))_w t^w = 1 + t^2 - t^{10} + t^{12} - t^{14} - t^{16} + t^{18} - 3t^{24} + 2t^{26}$$

$$+ t^{32} - 3t^{34} + 3t^{36} - 6t^{38} + 4t^{40} + t^{42} + t^{44} + 2t^{46} - t^{48} + 0t^{50} + \cdots$$

Here, our 0-dim cochain complex is $\mathbb{R}$, but in [3] that is $\{\mathbf{0}\}$.

5.2 How to use computer in order to get the dimension of relative cochain complex (brief summary)

1. For a given weight w, edit and fix the weight w in `../gkf_act-0.mpl` and run this maple script in order to get possible direct summands of $\mathrm{C}^{dp}_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{sp}(2,\mathbb{R}))_w$ for each degree dp.

   `% maple -q ../gkf_act-0.mpl > NotYet-`w

   Then, execute

   `% perl ../gkf_act-1.prl NotYet-`w

   and we get a file `OUT-`w.

2. Using the output file `OUT-`w, we run the next perl script

   `% perl ../gkf_act-2.prl OUT-`w

   After this job, we have a couple of output `range-`w and files `cases_w_dp.txt`.

   If we run

   `% perl ../mk-tex-gkf.prl OUT-`w

   we get a TEX-file `gkf_`w`.tex`. Compiling this, we get a table of degree, reference numbers and decompose type.

3. Key issue here is: First run perl,

   `% perl ../cooking.prl OUT-`w

   Then we get a output file `2eat_w.txt`

   In `../sophi.mpl`, (1) edit and fix the weight w, and (2) fix path to `range-w` and `2eat_w.txt`. Then run

   `% maple -q ../sophi.mpl > saving-`w`.txt`

   Then we see contribution of each type and the multiplicity of each cochain complex in `saving-`w`.txt`.

6 Getting concrete bases and matrix representation of d

Even though we know all the dimensions of subcomplex of relative cochain complex $C^m(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$, it is not enough to investigate the coboundary operator d itself. We have to know concrete bases of $C^m(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$ and $C^{m+1}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}(2n,\mathbb{R}))_w$, and the matrix representation of d with respect to those bases. In this section, we only deal with $n = 1$ and $M = \mathbb{R}^2$.
We follow the requirements for relative cochain σ. The first one is $i_\xi \sigma = 0$ ($\forall \xi \in \mathfrak{sp}(2,\mathbb{R})$). This is very easy to check σ contains $\mathfrak{S}_2$ and indeed, we already have omitted $\mathfrak{S}_2$. The other thing is $i_\xi d\,\sigma = 0$ ($\forall \xi \in \mathfrak{sp}(2,\mathbb{R})$). Since we have already observed that $i_\xi d = -\xi \cdot$, we get

$$\begin{aligned}(i_H \circ d)\, z^r_R &= -H \cdot z^r_R = -(2r - R) z^r_R \\ (i_X \circ d)\, z^r_R &= -X \cdot z^r_R = -(R - r) z^{r+1}_R \\ (i_Y \circ d)\, z^r_R &= -Y \cdot z^r_R - r z^{r-1}_R\end{aligned}$$

for each generators of 1-cochain complex. d is a skew-derivation of degree $+1$ and

$$d\, z^r_R = -\frac{r!(R-r)!}{2} \sum_{a+b=1+r, A+B=2+R} \begin{vmatrix} a & b \\ A & B \end{vmatrix} \frac{z^a_A}{a!(A-a)!} \wedge \frac{z^b_B}{b!(B-b)!}$$

where $0 \le r \le R$, $0 \le a \le A$, $0 \le b \le B$, $A \ge 2$, $B \ge 2$.

6.1 An easy exmaple

Here, we show a small practice of getting concrete basis of relative cochain complex of the case of $n = 1$ and the weight 6. As shown in Example 5.3 by some help of $Sp(2, \mathbb{R})$-theory, we have already known that $\dim C^2(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6 = \dim C^3(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6 = 1$ and the others are 0-dimensional, and furthermore the corresponding basis lives in $\Lambda^2\mathfrak{S}_5$ when $C^2(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6$, and lives in $\mathfrak{S}_3 \wedge \mathfrak{S}_4 \wedge \mathfrak{S}_5$ when $C^3(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6$. Let $\sigma = \displaystyle\sum_{0\leq i<j\leq 5} c_{i,j} z_5^i \wedge z_5^j$, and solve the equations $i_\xi \circ d\,\sigma = 0$ for $\xi \in \{X, Y, H\}$, which are a basis of $\mathfrak{sp}(2, \mathbb{R})$. Then we have

$$\sigma = c_{2,3}\left(10 z_5^0 \wedge z_5^5 - 2 z_5^1 \wedge z_5^4 + z_5^2 \wedge z_5^3\right)$$

and putting $c_{2,3} = 1$, we have σ_1, a basis of $C^2(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6$.

For 3-cochain, let $\tau = \displaystyle\sum_{i=0}^{3}\sum_{j=0}^{4}\sum_{k=0}^{5} c_{i,j,k} z_3^i \wedge z_4^j \wedge z_5^k$, and again solve the equations $i_\xi \circ d\,\sigma = 0$ for $\xi \in \{X, Y, H\}$. Then we have

$$\begin{aligned}\tau = c_{2,3,1}\big(& z_3^2 \wedge z_4^3 \wedge z_5^1 + 8 z_3^1 \wedge z_4^4 \wedge z_5^1 - 5/2 z_3^1 \wedge z_4^3 \wedge z_5^2 + z_3^1 \wedge z_4^2 \wedge z_5^3 - 6 z_3^0 \wedge z_4^2 \wedge z_5^4 \\ &+ 15 z_3^0 \wedge z_4^1 \wedge z_5^5 + z_3^1 \wedge z_4^1 \wedge z_5^4 - 20 z_3^1 \wedge z_4^0 \wedge z_5^5 - 6 z_3^0 \wedge z_4^4 \wedge z_5^2 + 9/2 z_3^0 \wedge z_4^3 \wedge z_5^3 \\ &+ 8 z_3^2 \wedge z_4^0 \wedge z_5^4 - 5/2 z_3^2 \wedge z_4^1 \wedge z_5^3 + z_3^2 \wedge z_4^2 \wedge z_5^2 - 20 z_3^2 \wedge z_4^4 \wedge z_5^0 - 6 z_3^3 \wedge z_4^0 \wedge z_5^3 \\ &+ 9/2 z_3^3 \wedge z_4^1 \wedge z_5^2 - 6 z_3^3 \wedge z_4^2 \wedge z_5^1 + 15 z_3^3 \wedge z_4^3 \wedge z_5^0\big)\end{aligned}$$

and putting $c_{2,3,1} = 4$, we have a basic vector τ_1 in $C^3(\mathfrak{ham}^0_2, \mathfrak{sp}(2, \mathbb{R}))_6$. We see that

$$d\,\sigma_1 = \tau_1$$

thus, we get $\mathrm{H}^2_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{s}p(2, \mathbb{R}))_6 = \{\mathbf{0}\}$, and so we have $\mathrm{H}^3_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{s}p(2, \mathbb{R}))_6 = \{\mathbf{0}\}$. Therefore, the Euler characteristic number of the alternating sum of the Betti numbers, which we include the trivial 0-th Betti number 1, is 1 as we have seen before.

6.2 More complicated example

By taking the example of weight = 8, we emphasize that how getting concrete basis is tough job. Later, we stress that our job sequences, which complete those jobs automatically, are how useful in this discussion.

deg	ref.#	type	dim
1	1	(10)	
2	1	$(3\ 9)$	
2	2	$(4\ 8)$	
2	3	$(5\ 7)$	
2	4	(6^2)	
3	1	$(3^2\ 8)$	
3	2	$(3\ 4\ 7)$	1
3	3	$(3\ 5\ 6)$	1
3	4	$(4^2\ 6)$	1
3	5	$(4\ 5^2)$	1

deg	ref.#	type	dim
4	1	$(3^3\ 7)$	
4	2	$(3^2\ 4\ 6)$	1
4	3	$(3^2\ 5^2)$	2
4	4	$(3\ 4^2\ 5)$	2
4	5	(4^4)	
5	1	$(3^4\ 6)$	
5	2	$(3^3\ 4\ 5)$	1
5	3	$(3^2\ 4^3)$	
6	1	$(3^4\ 4^2)$	

The cochain complex of deg = 3 is 4-dim, that of deg = 4 is 5-dim, and that of deg = 5 is 1-dimensional.
Here, we will see the whole process of getting a basis of each cochain complex.

1. 3-cochains

(a) type $(3,4,7)$-case: The candidate is $\sigma_1 = \displaystyle\sum_{i=0}^{3}\sum_{j=0}^{4}\sum_{k=0}^{7} c_{ijk} z_3^i \wedge z_4^j \wedge z_7^k$ with 160 unknown variables c_{ijk}. From the three conditions $i_\xi \circ d\,\sigma = 0$ for $\xi = X, Y, H$, we have a homogeneous linear equations (the number of equations is 455). Solving these equations, we have one undetermined parameter c_{007}, and we put it to be 1, and get a 1-dimensional basis:

(b) type $(3,5,6)$-case: The method is completely same and get a basis σ_2 by putting $c_{016} = 1$.

(c) type $(4^2,6)$-case: Here we have to be careful for dealing with exterior product elements as below. The candidate is $\sigma = \displaystyle\sum_{0\leq i<j\leq 4}\sum_{k=0}^{6} c_{ijk} z_4^i \wedge z_4^j \wedge z_6^k$ with 70 unknown variables c_{ijk}. Putting $c_{016} = 1$, we have a basis σ_3.

(d) type $(4,5^2)$-case: Here again we have to be careful for dealing with exterior product elements as before. The candidate is $\sigma_4 = \displaystyle\sum_{i=0}^{4}\sum_{0\leq j<k\leq 5} c_{ijk} z_4^i \wedge z_5^j \wedge z_5^k$ with 75 unknown variables c_{ijk}. Putting $c_{025} = 1$, we have a basis

$$
\begin{aligned}
\sigma_1 =& z_3^0 \wedge z_4^0 \wedge z_7^7 - 4z_3^0 \wedge z_4^1 \wedge z_7^6 + 6z_3^0 \wedge z_4^2 \wedge z_7^5 - 4z_3^0 \wedge z_4^3 \wedge z_7^4 + z_3^0 \wedge z_4^4 \wedge z_7^3 \\
&- 3z_3^1 \wedge z_4^0 \wedge z_7^6 + 12z_3^1 \wedge z_4^1 \wedge z_7^5 - 18z_3^1 \wedge z_4^2 \wedge z_7^4 + 12z_3^1 \wedge z_4^3 \wedge z_7^3 - 3z_3^1 \wedge z_4^4 \wedge z_7^2 \\
&+ 3z_3^2 \wedge z_4^0 \wedge z_7^5 - 12z_3^2 \wedge z_4^1 \wedge z_7^4 + 18z_3^2 \wedge z_4^2 \wedge z_7^3 - 12z_3^2 \wedge z_4^3 \wedge z_7^2 + 3z_3^2 \wedge z_4^4 \wedge z_7^1 \\
&- z_3^3 \wedge z_4^0 \wedge z_7^4 + 4z_3^3 \wedge z_4^1 \wedge z_7^3 - 6z_3^3 \wedge z_4^2 \wedge z_7^2 + 4z_3^3 \wedge z_4^3 \wedge z_7^1 - z_3^3 \wedge z_4^4 \wedge z_7^0 \\
\sigma_2 =& z_3^0 \wedge z_5^1 \wedge z_6^6 - 4z_3^0 \wedge z_5^2 \wedge z_6^5 + 6z_3^0 \wedge z_5^3 \wedge z_6^4 - 4z_3^0 \wedge z_5^4 \wedge z_6^3 + z_3^0 \wedge z_5^5 \wedge z_6^2 \\
&- z_3^1 \wedge z_5^0 \wedge z_6^6 + 2z_3^1 \wedge z_5^1 \wedge z_6^5 + 2z_3^1 \wedge z_5^2 \wedge z_6^4 - 8z_3^1 \wedge z_5^3 \wedge z_6^3 + 7z_3^1 \wedge z_5^4 \wedge z_6^2 \\
&- 2z_3^1 \wedge z_5^5 \wedge z_6^1 + 2z_3^2 \wedge z_5^0 \wedge z_6^5 - 7z_3^2 \wedge z_5^1 \wedge z_6^4 + 8z_3^2 \wedge z_5^2 \wedge z_6^3 - 2z_3^2 \wedge z_5^3 \wedge z_6^2 \\
&- 2z_3^2 \wedge z_5^4 \wedge z_6^1 + z_3^2 \wedge z_5^5 \wedge z_6^0 - z_3^3 \wedge z_5^0 \wedge z_6^4 + 4z_3^3 \wedge z_5^1 \wedge z_6^3 - 6z_3^3 \wedge z_5^2 \wedge z_6^2 \\
&+ 4z_3^3 \wedge z_5^3 \wedge z_6^1 - z_3^3 \wedge z_5^4 \wedge z_6^0 \\
\sigma_3 =& z_4^0 \wedge z_4^1 \wedge z_6^6 - 3z_4^0 \wedge z_4^2 \wedge z_6^5 + 3z_4^0 \wedge z_4^3 \wedge z_6^4 - z_4^0 \wedge z_4^4 \wedge z_6^3 + 6z_4^1 \wedge z_4^2 \wedge z_6^4 \\
&- 8z_4^1 \wedge z_4^3 \wedge z_6^3 + 3z_4^1 \wedge z_4^4 \wedge z_6^2 + 6z_4^2 \wedge z_4^3 \wedge z_6^2 - 3z_4^2 \wedge z_4^4 \wedge z_6^1 + z_4^3 \wedge z_4^4 \wedge z_6^0 \\
\sigma_4 =& z_4^0 \wedge z_5^2 \wedge z_5^5 - 3z_4^0 \wedge z_5^3 \wedge z_5^4 - 2z_4^1 \wedge z_5^1 \wedge z_5^5 + 4z_4^1 \wedge z_5^2 \wedge z_5^4 + z_4^2 \wedge z_5^0 \wedge z_5^5 \\
&+ z_4^2 \wedge z_5^1 \wedge z_5^4 - 8z_4^2 \wedge z_5^2 \wedge z_5^3 - 2z_4^3 \wedge z_5^0 \wedge z_5^4 + 4z_4^3 \wedge z_5^1 \wedge z_5^3 + z_4^4 \wedge z_5^0 \wedge z_5^3 \\
&- 3z_4^4 \wedge z_5^1 \wedge z_5^2
\end{aligned}
$$

Those $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are a basis of $C^3(\mathfrak{ham}_2^0, \mathfrak{sp}(2,\mathbb{R}))_8$.

2. 4-cochains

(a) type $(3^2,4,6)$-case: $c_{0206}\tau_1$, where τ_1 is below:

(b) type $(3^2,5^2)$-case: The general solution is $c_{0314}\tau_2 + c_{0314}\tau_3$, where τ_2, τ_3 are below:

(c) type $(3,4^2,5)$-case: The general solution is $c_{0125}\tau_4 + c_{0035}\tau_5$, where τ_4, τ_5 are below:

$$
\begin{aligned}
\tau_1 =& - 2z_3^0 \wedge z_3^1 \wedge z_4^1 \wedge z_6^6 + 6z_3^0 \wedge z_3^1 \wedge z_4^2 \wedge z_6^5 - 6z_3^0 \wedge z_3^1 \wedge z_4^3 \wedge z_6^4 + 2z_3^0 \wedge z_3^1 \wedge z_4^4 \wedge z_6^3 \\
&+ z_3^0 \wedge z_3^2 \wedge z_4^0 \wedge z_6^6 - 6z_3^0 \wedge z_3^2 \wedge z_4^2 \wedge z_6^4 + 8z_3^0 \wedge z_3^2 \wedge z_4^3 \wedge z_6^3 - 3z_3^0 \wedge z_3^2 \wedge z_4^4 \wedge z_6^2 \\
&- z_3^0 \wedge z_3^3 \wedge z_4^0 \wedge z_6^5 + 2z_3^0 \wedge z_3^3 \wedge z_4^1 \wedge z_6^4 - 2z_3^0 \wedge z_3^3 \wedge z_4^3 \wedge z_6^2 + z_3^0 \wedge z_3^3 \wedge z_4^4 \wedge z_6^1 \\
&- 3z_3^1 \wedge z_3^2 \wedge z_4^0 \wedge z_6^5 + 6z_3^1 \wedge z_3^2 \wedge z_4^1 \wedge z_6^4 - 6z_3^1 \wedge z_3^2 \wedge z_4^3 \wedge z_6^2 + 3z_3^1 \wedge z_3^2 \wedge z_4^4 \wedge z_6^1
\end{aligned}
$$

$$
\begin{aligned}
&+ 3z_3^1 \wedge z_3^3 \wedge z_4^0 \wedge z_6^4 - 8z_3^1 \wedge z_3^3 \wedge z_4^1 \wedge z_6^3 + 6z_3^1 \wedge z_3^3 \wedge z_4^2 \wedge z_6^2 - z_3^1 \wedge z_3^3 \wedge z_4^4 \wedge z_6^0 \\
&- 2z_3^2 \wedge z_3^3 \wedge z_4^0 \wedge z_6^3 + 6z_3^2 \wedge z_3^3 \wedge z_4^1 \wedge z_6^2 - 6z_3^2 \wedge z_3^3 \wedge z_4^2 \wedge z_6^1 + 2z_3^2 \wedge z_3^3 \wedge z_4^3 \wedge z_6^0 \\
\tau_2 =& 3z_3^0 \wedge z_3^1 \wedge z_5^2 \wedge z_5^5 - 9z_3^0 \wedge z_3^1 \wedge z_5^3 \wedge z_5^4 - 3z_3^0 \wedge z_3^2 \wedge z_5^1 \wedge z_5^5 + 6z_3^0 \wedge z_3^2 \wedge z_5^2 \wedge z_5^4 \\
&+ \frac{2}{5}z_3^0 \wedge z_3^3 \wedge z_5^0 \wedge z_5^5 + z_3^0 \wedge z_3^3 \wedge z_5^1 \wedge z_5^4 - 5z_3^0 \wedge z_3^3 \wedge z_5^2 \wedge z_5^3 + \frac{9}{5}z_3^1 \wedge z_3^2 \wedge z_5^0 \wedge z_5^5 \\
&- 9z_3^1 \wedge z_3^2 \wedge z_5^2 \wedge z_5^3 - 3z_3^1 \wedge z_3^3 \wedge z_5^0 \wedge z_5^4 + 6z_3^1 \wedge z_3^3 \wedge z_5^1 \wedge z_5^3 + 3z_3^2 \wedge z_3^3 \wedge z_5^0 \wedge z_5^3 \\
&- 9z_3^2 \wedge z_3^3 \wedge z_5^1 \wedge z_5^2 \\
\tau_3 =& z_3^0 \wedge z_3^1 \wedge z_5^2 \wedge z_5^5 - 3z_3^0 \wedge z_3^1 \wedge z_5^3 \wedge z_5^4 - z_3^0 \wedge z_3^2 \wedge z_5^1 \wedge z_5^5 + 2z_3^0 \wedge z_3^2 \wedge z_5^2 \wedge z_5^4 \\
&+ \frac{1}{5}z_3^0 \wedge z_3^3 \wedge z_5^0 \wedge z_5^5 - z_3^0 \wedge z_3^3 \wedge z_5^2 \wedge z_5^3 + \frac{2}{5}z_3^1 \wedge z_3^2 \wedge z_5^0 \wedge z_5^5 + z_3^1 \wedge z_3^2 \wedge z_5^1 \wedge z_5^4 \\
&- 5z_3^1 \wedge z_3^2 \wedge z_5^2 \wedge z_5^3 - z_3^1 \wedge z_3^3 \wedge z_5^0 \wedge z_5^4 + 2z_3^1 \wedge z_3^3 \wedge z_5^1 \wedge z_5^3 + z_3^2 \wedge z_3^3 \wedge z_5^0 \wedge z_5^3 \\
&- 3z_3^2 \wedge z_3^3 \wedge z_5^1 \wedge z_5^2 \\
\tau_4 =& -\frac{2}{3}z_3^0 \wedge z_4^0 \wedge z_4^3 \wedge z_5^5 + \frac{2}{3}z_3^0 \wedge z_4^0 \wedge z_4^4 \wedge z_5^4 + z_3^0 \wedge z_4^1 \wedge z_4^2 \wedge z_5^5 + \frac{2}{3}z_3^0 \wedge z_4^1 \wedge z_4^3 \wedge z_5^4 \\
&- \frac{5}{3}z_3^0 \wedge z_4^1 \wedge z_4^4 \wedge z_5^3 - z_3^0 \wedge z_4^2 \wedge z_4^3 \wedge z_5^3 + 2z_3^0 \wedge z_4^2 \wedge z_4^4 \wedge z_5^2 - z_3^0 \wedge z_4^3 \wedge z_4^4 \wedge z_5^1 \\
&+ z_3^1 \wedge z_4^0 \wedge z_4^2 \wedge z_5^5 - z_3^1 \wedge z_4^0 \wedge z_4^4 \wedge z_5^3 - 7z_3^1 \wedge z_4^1 \wedge z_4^2 \wedge z_5^4 + 6z_3^1 \wedge z_4^1 \wedge z_4^3 \wedge z_5^3 \\
&+ z_3^1 \wedge z_4^1 \wedge z_4^4 \wedge z_5^2 - z_3^1 \wedge z_4^2 \wedge z_4^4 \wedge z_5^1 - 5z_3^1 \wedge z_4^2 \wedge z_4^3 \wedge z_5^2 + z_3^1 \wedge z_4^3 \wedge z_4^4 \wedge z_5^0 \\
&- z_3^2 \wedge z_4^0 \wedge z_4^1 \wedge z_5^5 + z_3^2 \wedge z_4^0 \wedge z_4^2 \wedge z_5^4 + z_3^2 \wedge z_4^0 \wedge z_4^4 \wedge z_5^2 - z_3^2 \wedge z_4^0 \wedge z_4^3 \wedge z_5^3 \\
&+ 5z_3^2 \wedge z_4^1 \wedge z_4^2 \wedge z_5^3 - 6z_3^2 \wedge z_4^1 \wedge z_4^3 \wedge z_5^2 + 7z_3^2 \wedge z_4^2 \wedge z_4^3 \wedge z_5^1 - z_3^2 \wedge z_4^2 \wedge z_4^4 \wedge z_5^0 \\
&+ z_3^3 \wedge z_4^0 \wedge z_4^1 \wedge z_5^4 - 2z_3^3 \wedge z_4^0 \wedge z_4^2 \wedge z_5^3 + \frac{5}{3}z_3^3 \wedge z_4^0 \wedge z_4^3 \wedge z_5^2 - \frac{2}{3}z_3^3 \wedge z_4^0 \wedge z_4^4 \wedge z_5^1 \\
&+ z_3^3 \wedge z_4^1 \wedge z_4^2 \wedge z_5^2 - \frac{2}{3}z_3^3 \wedge z_4^1 \wedge z_4^3 \wedge z_5^1 + \frac{2}{3}z_3^3 \wedge z_4^1 \wedge z_4^4 \wedge z_5^0 - z_3^3 \wedge z_4^2 \wedge z_4^3 \wedge z_5^0 \\
\tau_5 =& z_3^0 \wedge z_4^0 \wedge z_4^3 \wedge z_5^5 - z_3^0 \wedge z_4^0 \wedge z_4^4 \wedge z_5^4 - 4z_3^0 \wedge z_4^1 \wedge z_4^3 \wedge z_5^4 + 4z_3^0 \wedge z_4^1 \wedge z_4^4 \wedge z_5^3 \\
&+ 6z_3^0 \wedge z_4^2 \wedge z_4^3 \wedge z_5^3 - 6z_3^0 \wedge z_4^2 \wedge z_4^4 \wedge z_5^2 + 3z_3^0 \wedge z_4^3 \wedge z_4^4 \wedge z_5^1 - 3z_3^1 \wedge z_4^0 \wedge z_4^2 \wedge z_5^5 \\
&+ 12z_3^1 \wedge z_4^1 \wedge z_4^2 \wedge z_5^4 - 12z_3^1 \wedge z_4^1 \wedge z_4^3 \wedge z_5^3 + 3z_3^1 \wedge z_4^2 \wedge z_4^4 \wedge z_5^1 + 6z_3^1 \wedge z_4^2 \wedge z_4^3 \wedge z_5^2 \\
&- 3z_3^1 \wedge z_4^3 \wedge z_4^4 \wedge z_5^0 + 3z_3^2 \wedge z_4^0 \wedge z_4^1 \wedge z_5^5 - 3z_3^2 \wedge z_4^0 \wedge z_4^2 \wedge z_5^4 - 6z_3^2 \wedge z_4^1 \wedge z_4^2 \wedge z_5^3 \\
&+ 12z_3^2 \wedge z_4^1 \wedge z_4^3 \wedge z_5^2 - 12z_3^2 \wedge z_4^2 \wedge z_4^3 \wedge z_5^1 + 3z_3^2 \wedge z_4^2 \wedge z_4^4 \wedge z_5^0 - 3z_3^3 \wedge z_4^0 \wedge z_4^1 \wedge z_5^4 \\
&+ 6z_3^3 \wedge z_4^0 \wedge z_4^2 \wedge z_5^3 - 4z_3^3 \wedge z_4^0 \wedge z_4^3 \wedge z_5^2 + z_3^3 \wedge z_4^0 \wedge z_4^4 \wedge z_5^1 - 6z_3^3 \wedge z_4^1 \wedge z_4^2 \wedge z_5^2 \\
&+ 4z_3^3 \wedge z_4^1 \wedge z_4^3 \wedge z_5^1 - z_3^3 \wedge z_4^1 \wedge z_4^4 \wedge z_5^0 + 3z_3^1 \wedge z_4^0 \wedge z_4^3 \wedge z_5^4 - 3z_3^2 \wedge z_4^1 \wedge z_4^4 \wedge z_5^1
\end{aligned}
$$

3. 5-cochain

   type (3^3 4 5)-case: 120 unknown variables and solving the linear 339-equations, and putting $c_{01314} = 1$, we have a basis:

$$
\begin{aligned}
\rho =& 3z_3^0 \wedge z_3^1 \wedge z_3^2 \wedge z_4^1 \wedge z_5^5 - 9z_3^0 \wedge z_3^1 \wedge z_3^2 \wedge z_4^2 \wedge z_5^4 + 9z_3^0 \wedge z_3^1 \wedge z_3^2 \wedge z_4^3 \wedge z_5^3 \\
&- 3z_3^0 \wedge z_3^1 \wedge z_3^2 \wedge z_4^4 \wedge z_5^2 - z_3^0 \wedge z_3^1 \wedge z_3^3 \wedge z_4^0 \wedge z_5^5 + z_3^0 \wedge z_3^1 \wedge z_3^3 \wedge z_4^1 \wedge z_5^4 \\
&+ 3z_3^0 \wedge z_3^1 \wedge z_3^3 \wedge z_4^2 \wedge z_5^3 - 5z_3^0 \wedge z_3^1 \wedge z_3^3 \wedge z_4^3 \wedge z_5^2 + 2z_3^0 \wedge z_3^1 \wedge z_3^3 \wedge z_4^4 \wedge z_5^1 \\
&+ 2z_3^0 \wedge z_3^2 \wedge z_3^3 \wedge z_4^0 \wedge z_5^4 - 5z_3^0 \wedge z_3^2 \wedge z_3^3 \wedge z_4^1 \wedge z_5^3 + 3z_3^0 \wedge z_3^2 \wedge z_3^3 \wedge z_4^2 \wedge z_5^2 \\
&+ z_3^0 \wedge z_3^2 \wedge z_3^3 \wedge z_4^3 \wedge z_5^1 - z_3^0 \wedge z_3^2 \wedge z_3^3 \wedge z_4^4 \wedge z_5^0 - 3z_3^1 \wedge z_3^2 \wedge z_3^3 \wedge z_4^0 \wedge z_5^3 \\
&+ 9z_3^1 \wedge z_3^2 \wedge z_3^3 \wedge z_4^1 \wedge z_5^2 - 9z_3^1 \wedge z_3^2 \wedge z_3^3 \wedge z_4^2 \wedge z_5^1 + 3z_3^1 \wedge z_3^2 \wedge z_3^3 \wedge z_4^3 \wedge z_5^0
\end{aligned}
$$

With respect to those concrete bases of relative cochain complex, the coboundary operator d between degree 4 and 5, is of the form $d\,\tau_j = b_j\rho \quad (j = 1..5)$ and comparing both sides, we get a matrix expression $B := \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix} = \begin{pmatrix} -4 & -3 & -3 & -9 & 6 \end{pmatrix}$.

About the coboundary operator d between degree 3 and 4, $d\,\sigma_j = \sum_{i=1}^{5} a_j^i \tau_i$ $(j = 1, \ldots, 4)$, For each i, comparing both sides again, we get a matrix expression

$$A = \left(a_i^j\right) = \begin{pmatrix} -6 & -12 & \frac{9}{4} & 0 \\ 0 & -27 & 0 & 1 \\ 0 & 59 & 0 & 3 \\ 5 & 4 & \frac{3}{4} & 22 \\ \frac{7}{2} & 14 & \frac{21}{8} & 35 \end{pmatrix}$$

As expected, we see that $BA = O$, which corresponds to $d \circ d = 0$. Since rank$A = 4$, rank$B = 1$, thus we have $\mathrm{H}^3_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{sp}(2,\mathbb{R}))_8 = \{\mathbf{0}\}$, $\mathrm{H}^4_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{sp}(2,\mathbb{R}))_8 = \{\mathbf{0}\}$, $\mathrm{H}^5_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{sp}(2,\mathbb{R}))_8 = \{\mathbf{0}\}$. These mean that except $\mathrm{H}^0_{\mathrm{GF}}(\mathfrak{h}am^0_2, \mathfrak{sp}(2,\mathbb{R}))_8 = \mathbb{R}$, the other cohomologies are zero and the Euler characteristic number is 1.

6.3 How tuning computer to subscribe our job

We here show our typical strategy for getting a concrete basis of relative cochain complex. Take an example with weight 8, which we have seen above. We follow the above discussion in the case of type $(3, 5, 6)$ and type $(4^2, 6)$ in order to emphasize some difference. We need to distinguish scalar and vectors in general. For that purpose, we use `difforms` package. In the type $(3, 5, 7)$ of degree 3 with weight 8. a cochain in general is

$$\sigma = \sum_{i_1=0}^{3} \sum_{i_2=0}^{5} \sum_{i_3=0}^{7} c_{i_1,i_2,i_3} z_3^{i_1} \otimes z_5^{i_2} \otimes z_7^{i_3}$$

(in this case, $\wedge$ and $\otimes$ have the same meaning), and we declare c_{i_1,i_2,i_3} are scalar. For that purpose we prepare `in_8_3-3_a.txt`.

```
uke31 := NULL: uke32 := NULL:
for i1 from 0 to d3 do
for i2 from 0 to d5 do
for i3 from 0 to d6 do
   uke31 := uke31, cat(c3, "_",i1, "_",i2, "_",i3) = const ;
   uke32 := uke32, cat(c3, "_",i1, "_",i2, "_",i3) ;
od od od:
defform(uke31):
```

To construct σ above, we prepare `in_8_3-3_b.txt`.

```
# defform( A3 = 3): A3 :=0:
for i1 from 0 to d3 do
for i2 from 0 to d5 do
for i3 from 0 to d6 do
A3 := A3 + cat(c3, "_",i1, "_",i2, "_",i3) *
&^ ( z[i1,d3-i1], z[i2,d5-i2], z[i3,d6-i3] ):
od od od:
```

In the type $(4^2, 6)$ of degree 3 with weight 8. a general cochain is

$$\sigma = \sum_{i_1=0}^{4} \sum_{i_2=0}^{4} \sum_{i_3=0}^{6} c_{i_1,i_2,i_3} z_4^{i_1} \wedge z_4^{i_2} \wedge z_6^{i_3}, \quad \text{where } c_{i_1,i_2,i_3} \text{ are skewsymmetric in 3 indices.}$$

To avoid complicated requirement, we restrict range of indices as

$$\sigma = \sum_{0 \leq i_1 < i_2 \leq 4} \sum_{i_3=0}^{6} c_{i_1,i_2,i_3} z_4^{i_1} \wedge z_4^{i_2} \otimes z_6^{i_3},$$

(we need some scale factor, in the case above, 2-times, but we ignore them here after.)
This time, a declaration that coefficients are constant, is `difforms`, we prepare `in_8_3-4_a.txt`.

```
uke41 := NULL: uke42 := NULL:
for i1 from 0 to d4 do
for i2 from 1+i1 to d4 do
for i3 from 0 to d6 do
   uke41 := uke41, cat(c4, "_",i1, "_",i2, "_",i3) = const ;
   uke42 := uke42, cat(c4, "_",i1, "_",i2, "_",i3) ;
od od od:
defform(uke41):
```

To construct σ above, we prepare in_8_3-4_b.txt.

```
# defform( A4 = 3): A4 :=0:
for i1 from 0 to d4 do
for i2 from 1+i1 to d4 do
for i3 from 0 to d6 do
A4 := A4 + cat(c4, "_",i1, "_",i2, "_",i3) *
&^ ( z[i1,d4-i1], z[i2,d4-i2], z[i3,d6-i3] ):
od od od:
```

To pick up "coefficients" of $i_\xi d(\sigma)$ $(\forall \xi \in \mathfrak{sp}(2,\mathbb{R}))$ with respect "some generators", we prepare in_8_3-4_c.txt.

```
for i1 from 0 to d4 do
for i2 from 1+i1 to d4 do
for i3 from 0 to d6 do
ukez := ukez, seq(
mInnd([z[i1,d4-i1],z[i2,d4-i2],z[i3,d6-i3]],
                          W_W[s]),s=0..d0);
od od od:
```

For the main purpose to solve $i_\xi d(\sigma) = 0$ $(\forall \xi \in \mathfrak{sp}(2,\mathbb{R}))$ and to determine σ and $d(\sigma)$, we have a core Maple script action_new.mpl, we omit it because it will occupy one and a half pages.

Remark 6.1 When $w \geq 16$, we encounter some trouble of kernel panic in processing the above steps. Main problem seems shortage of CPU memory. To recover this trouble, we change our process slightly. A big guarantee is *linearity* of d. Namely, suppose for a given "huge " cochain A we cannot compute $d(A)$. Then, we divide A into "small" pieces like $A = a_1 + a_2 + \cdots$. Instead of handling $d(A)$ itself, we manipulate $d(a_i)$, and by $d(a_1) + d(a_2) + \cdots$, we get the whole $d(A)$.

6.4 Sequence of computer process to get concrete basis of cochain complex (brief summary)

1. The same (1) of §5.2.

2. The same (2) of §5.2.

3. We prepare a plain text file in_dp-new.txt, this is a prototype of cochain of degree dp, indep of w. In which we prepare coefficients of degree dp cochain, and ready to handle to defform() of difforms Maple package, and (Of course, the range of dp depends on w.)

   We manipulate this in_dp-new.txt and cases_w_dp.txt in step 2 by the perl script gkf_act-3.prl.

   % perl ../gkf_act-3.prl

   Out put file are out_w_dp-ref#.

4. **Important!** For out_w_dp-ref#, we have to revise it in order to guarantee the skew-symmetry. For instance, the right side is desired form:

for i1 from 0 to d4 do	for i1 from 0 to d4 do
for i2 from 0 to d4 do	for i2 from <u>1+i1</u> to d4 do

For that purpose, we prepare a perl script `gkf_act-4.prl` and run for all files:

```
% ../gkf_act-4.prl out_w_dp-ref#
```

Output are `out_w_dp-ref#.txt`.

5. For the revised `out_w_dp-ref#.txt`, we apply `gkf_act-5.prl`.

   ```
   % ../gkf_act-5.prl out_w_dp-ref#.txt
   ```

   Output are `in_w_dp-ref#_[abc].txt`. In §6.2, we encountered with `in_8_3-4_[abc].txt`.

6. We prepare a maple script `action_new.mpl`, in which for fixed weight w and degree dp, get a basis of cocycles and d-image of them for each direct sum componet labeled by ref#.

   We omitted to explain about this key job in §6.2.

7 Main result

While weight is less that 12, the whole cohomology groups were studied in [3].

Theorem 2 Using those simple but powerful tricks explained above, we could finish all computations for weight from 12 to 18. We will show the results a table below. The abbreviations in the table mean degree k is $\mathtt{Sp}(2,\mathbb{R})$-invariant cochain complex C^k, dim is dimC^k, and rank(d) is the rank of $d : C^k \longrightarrow C^{1+k}$.

weight	degree	1	2	3	4	5	6	7	8	9	10
12	dim	0	0	8	23	22	13	5	0	0	0
	rank(d)	0	0	8	14	8	5	0	0	0	0
	Betti#	0	0	0	1	0	0	0	0	0	0
14	dim	0	1	6	31	71	58	15	2	1	0
	rank(d)	0	1	5	26	44	14	1	1	0	0
	Betti#	0	0	0	0	1	0	0	0	0	0
16	dim	0	0	12	61	126	147	95	24	0	0
	rank(d)	0	0	12	49	77	70	24	0	0	0
	Betti#	0	0	0	0	0	0	1	0	0	0
18	dim	0	1	10	80	262	380	268	100	21	1
	rank(d)	0	1	9	71	191	188	80	20	1	0
	Betti#	0	0	0	0	0	1	0	0	0	0

Remark 7.1 The Euler characteristic number is the alternating sum of *dim* of cochain complexes or Betti numbers, including 0-dimensional. The tables above show that $\chi(\text{weight} = 12) = 2$, $\chi(\text{weight} = 14) = 0$, $\chi(\text{weight} = 16) = 0$, and $\chi(\text{weight} = 18) = 2$.
When the weight = 20, we have the complete list of degree of all relative cochain complexes as below:

degree	1	2	3	4	5	6	7	8	9	10	11
dim	0	0	17	124	423	791	801	414	96	9	1

and the Euler characteristic number, i.e., the alternating sum including 0-dimensional, is 1. It will be interesting to check if there are non-trivial cohomology groups with weight 20.

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